IN - CLASS EXERCISES

1. Find $(1 + \alpha_1 L + \alpha_2 L^2)^{-1}$ up to and including the term in L^3 (ψ_3).

$\alpha_0 \psi_0 = \delta_0$

$\alpha_0 \psi_1 + \alpha_1 \psi_0 = \delta_1$

$\alpha_0 \psi_2 + \alpha_1 \psi_1 + \alpha_2 \psi_0 = \delta_2$

$\alpha_0 \psi_3 + \alpha_1 \psi_2 + \alpha_2 \psi_1 + \alpha_3 \psi_0 = \delta_3$

$\vdots$

here, $\alpha(L)\,\psi(L) = \delta(L)$;

$\delta_0 = 1;\ \delta_1 = 0;\ \delta_2 = 0;\ \ldots$

$\alpha_0 = 1;\ \alpha_1 = \alpha_1;\ \alpha_2 = \alpha_2;\ \alpha_3 = 0;\ \alpha_4 = 0; \ldots$

$\psi_0 = 1$

$\psi_1 + \alpha_1 \psi_0 = 0$

recursion $\left(\begin{array}{l} \psi_2 + \alpha_1 \psi_1 + \alpha_2 \psi_0 = 0 \\ \psi_3 + \alpha_1 \psi_2 + \alpha_2 \psi_1 = 0 \end{array} \right.$

$\vdots$

$\psi_0 = 1$

$\psi_1 = -\alpha_1 \psi_0 = -\alpha_1$

$\psi_2 = -\alpha_1 \psi_1 - \alpha_2 \psi_0 = \alpha_1^2 - \alpha_2$

$\psi_3 = -\alpha_1 \psi_2 - \alpha_2 \psi_1 = -\alpha_1(\alpha_1^2 - \alpha_2) + \alpha_1 \alpha_2$

$= -\alpha_1^3 + 2\alpha_1 \alpha_2$

so $\psi(L) := \alpha(L)^{-1} = \underline{1 - \alpha_1 L + (\alpha_1^2 - \alpha_2) L^2 - (\alpha_1^3 - 2\alpha_1\alpha_2) L^3 + \cdots}$

2. Does $\alpha(z) = 1 - \frac{3}{4} z + \frac{9}{16} z^2$ satisfy the stability condition?

Roots λ of $\alpha^*(\lambda) = \lambda^2 - \frac{3}{4}\lambda + \frac{9}{16} = 0$ must lie inside the unit circle

Immediately, $\lambda = \frac{1}{2}\left(\frac{3}{4} \pm \sqrt{\frac{9}{16} - 4\left(\frac{9}{16}\right)}\right) = \frac{1}{2}\left(\frac{3}{4} \pm \sqrt{-3\left(\frac{9}{16}\right)}\right)$ and so λ_1 and λ_2 are complex.

Let $\lambda = h + vi$ $(i = \sqrt{-1})$. $\alpha^*(\lambda)$ gives $(h+vi)^2 - \frac{3}{4}(h+vi) + \frac{9}{16} = 0$.

So, $h^2 + 2hvi - v^2 - \frac{3}{4}h - \frac{3}{4}vi + \frac{9}{16} = 0$

$\Rightarrow\ h^2 - \frac{3}{4}h + \frac{9}{16} = v^2 \qquad (1)$

$(2h - \frac{3}{4})v = 0 \qquad (2).$

From (2), $v = 0$ or $2h - \frac{3}{4} = 0$.

If $v = 0$, then (1) becomes $h^2 - \frac{3}{4}h + \frac{9}{16} = 0$, which is of the same form as $\alpha^*(\lambda) = 0$; and so we have made no progress!

So, if $2h - \frac{3}{4} = 0 \Rightarrow h = \frac{3}{8}$ then (1) gives $\left(\frac{3}{8}\right)^2 - \frac{3}{4}\left(\frac{3}{8}\right) + \frac{9}{16} = v^2$

$$\Rightarrow \frac{9}{64} - \frac{9}{32} + \frac{9}{16} = v^2 \Rightarrow v^2 = \frac{27}{64}$$

Then, $|\lambda| = |h + vi| = \sqrt{h^2 + v^2} = \sqrt{\frac{9}{64} + \frac{27}{64}} = \sqrt{\frac{36}{64}} = \frac{6}{8} = \underline{\frac{3}{4} < 1}$

and so <u>$\alpha(z)$ satisfies the stability condition</u>.

3. <u>Find ρ_s for the MA(2) without drift</u>: $y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2}$; $\varepsilon_t \sim iid(0, \sigma^2_\varepsilon)$.

$$E(y_t) = 0$$

$$Var(y_t) = E(y_t^2) = E[(\varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2})(\varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2})]$$

$$= E[\varepsilon_t^2 + \beta_1^2 \varepsilon_{t-1}^2 + \beta_2^2 \varepsilon_{t-2}^2 + \text{cross-products}]$$

$$= (1 + \beta_1^2 + \beta_2^2)\sigma^2_\varepsilon$$

$$Cov(y_t, y_{t-1}) = E(y_t y_{t-1}) = E[(\varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2})(\varepsilon_{t-1} + \beta_1 \varepsilon_{t-2} + \beta_2 \varepsilon_{t-3})]$$

$$= \beta_1 \sigma^2_\varepsilon + \beta_1 \beta_2 \sigma^2_\varepsilon = \beta_1 (1 + \beta_2)\sigma^2_\varepsilon$$

$$Cov(y_t, y_{t-2}) = E(y_t y_{t-2}) = E[(\varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2})(\varepsilon_{t-2} + \beta_1 \varepsilon_{t-3} + \beta_2 \varepsilon_{t-4})]$$

$$= \beta_2 \sigma^2_\varepsilon$$

$Cov(y_t, y_{t-3}) = 0$ etc.

$$\rho_s = \frac{Cov(y_t, y_{t-s})}{\sqrt{Var(y_t)}\sqrt{Var(y_{t-s})}} = \begin{cases} 1 & ; \quad s = 0 \\ \dfrac{\beta_1(1+\beta_2)}{1 + \beta_1^2 + \beta_2^2} & ; \quad s = 1 \\ \dfrac{\beta_2}{1 + \beta_1^2 + \beta_2^2} & ; \quad s = 2 \\ 0 & ; \quad s = 3,\ s = 4, \ldots \end{cases}$$

4. Find ρ_s for the ARMA(1,1) without drift: (THIS QUESTION IS HARD! — AND WOULD NOT APPEAR ON THE EXAM...)

$$y_t = \alpha_1 y_{t-1} + \varepsilon_t + \beta_1 \varepsilon_{t-1}; \quad \varepsilon_t \sim iid(0, \sigma^2_\varepsilon). \qquad (1)$$

Given stationarity, $y_t := \mu$ (and if $\alpha_1 \neq 0$, so that ARMA(1,1) $\not\to$ MA(1))

Then $E(y_t) = \alpha_1 E(y_{t-1})$ so $\mu = \alpha_1 \mu$ and $\mu = 0$

Alternatively, $(1-\alpha_1 L) y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1}$ and $y_t = (1-\alpha_1 L)^{-1}(\varepsilon_t + \beta_1 \varepsilon_{t-1})$

from which $E(y_t) = 0$ immediately.

Define $Cov(y_t, y_{t-s}) = E[y_t y_{t-s}] =: \gamma_s$

Multiply both sides of (1) by y_{t-s}:

$$y_t y_{t-s} = \alpha_1 y_{t-1} y_{t-s} + \varepsilon_t y_{t-s} + \beta_1 \varepsilon_{t-1} y_{t-s}$$

For $s = 2, 3, 4, \ldots$ y_{t-s} is not correlated with ε_{t-1} (nor with ε_t), so

$$E(y_t y_{t-s}) = \alpha_1 E(y_{t-1} y_{t-s}) \Rightarrow \gamma_s = \alpha_1 \gamma_{s-1} \quad \text{(recursion)}$$

$$\Rightarrow \rho_s = \alpha_1 \rho_{s-1}$$

For $s = 0, 1$:

$$y_t = (1-\alpha_1 L)^{-1}(\varepsilon_t + \beta_1 \varepsilon_{t-1})$$

$$= (1 + \alpha_1 L + \alpha_1^2 L^2 + \alpha_1^3 L^3 + \ldots)(\varepsilon_t + \beta_1 \varepsilon_{t-1})$$

$$= (\varepsilon_t + \alpha_1 \varepsilon_{t-1} + \alpha_1^2 \varepsilon_{t-2} + \alpha_1^3 \varepsilon_{t-3} + \ldots)$$
$$+ \beta_1 (\varepsilon_{t-1} + \alpha_1 \varepsilon_{t-2} + \alpha_1^2 \varepsilon_{t-3} + \alpha_1^3 \varepsilon_{t-4} + \ldots)$$

$$= \varepsilon_t + (\alpha_1 + \beta_1)\varepsilon_{t-1} + (\alpha_1\beta_1 + \alpha_1^2)\varepsilon_{t-2} + (\alpha_1^2\beta_1 + \alpha_1^3)\varepsilon_{t-3} + \ldots$$

$$= \varepsilon_t + (\alpha_1 + \beta_1)\varepsilon_{t-1} + \alpha_1(\alpha_1 + \beta_1)\varepsilon_{t-2} + \alpha_1^2(\alpha_1 + \beta_1)\varepsilon_{t-3} + \ldots$$

$$\Rightarrow y_t^2 = \varepsilon_t^2 + (\alpha_1 + \beta_1)^2 \varepsilon_{t-1}^2 + \alpha_1^2(\alpha_1 + \beta_1)^2 \varepsilon_{t-2}^2 + \alpha_1^4(\alpha_1 + \beta_1)^2 \varepsilon_{t-3}^2 + \ldots$$

+ cross-products

$$\gamma_0 = E(y_t^2) = [1 + (\alpha_1 + \beta_1)^2 + \alpha_1^2(\alpha_1 + \beta_1)^2 + \alpha_1^4(\alpha_1 + \beta_1)^2 + \ldots]\sigma^2_\varepsilon$$

$$= \sigma^2_\varepsilon + (\alpha_1 + \beta_1)^2 [1 + \alpha_1^2 + \alpha_1^4 + \ldots]\sigma^2_\varepsilon$$

$$= \sigma^2_\varepsilon \left(1 + \frac{(\alpha_1 + \beta_1)^2}{1 - \alpha_1^2}\right)$$

$$\gamma_1 = E(y_t y_{t-1}) = E\Big[\{\varepsilon_t + (\alpha_1+\beta_1)\varepsilon_{t-1} + \alpha_1(\alpha_1+\beta_1)\varepsilon_{t-2} + \alpha_1^2(\alpha_1+\beta_1)\varepsilon_{t-3} + \\ \{\varepsilon_{t-1} + (\alpha_1+\beta_1)\varepsilon_{t-2} + \alpha_1(\alpha_1+\beta_1)\varepsilon_{t-3} + \alpha_1^2(\alpha_1+\beta_1)\varepsilon_{t-4}\ldots$$

$$= E\left[(\alpha_1+\beta_1)\varepsilon_{t-1}^2 + \alpha_1(\alpha_1+\beta_1)^2\varepsilon_{t-2}^2 + \alpha_1^3(\alpha_1+\beta_1)^2\varepsilon_{t-3}^2 + \ldots\right]$$

$$= (\alpha_1+\beta_1)\,E\left[\varepsilon_{t-1}^2 + \alpha_1(\alpha_1+\beta_1)\varepsilon_{t-2}^2 + \alpha_1^3(\alpha_1+\beta_1)\varepsilon_{t-3} + \ldots\right]$$

$$= (\alpha_1+\beta_1)\left[\sigma^2_\varepsilon + \alpha_1(\alpha_1+\beta_1)\left[\sigma^2_\varepsilon(1+\alpha_1^2+\alpha_1^4+\ldots)\right]\right]$$

$$= (\alpha_1+\beta_1)\left[\sigma^2_\varepsilon\left[1 + \frac{\alpha_1(\alpha_1+\beta_1)}{1-\alpha_1^2}\right]\right]$$

i.e.,
$$\rho_s = \begin{cases} 1 & ; \quad s = 0 \\[2ex] \dfrac{(\alpha_1+\beta_1)\left[1 + \dfrac{\alpha_1(\alpha_1+\beta_1)}{1-\alpha_1^2}\right]}{1 + \dfrac{(\alpha_1+\beta_1)^2}{1-\alpha_1^2}} & ; \quad s = 1 \end{cases}$$

and
$$\begin{cases} \rho_2 = \alpha_1 \rho_1 \\ \rho_3 = \alpha_1 \rho_2 = \alpha_1^2 \rho_1 \\ \vdots \\ (\rho_s = \alpha_1^s \rho_{s-1}) \; ; \quad s \geq 2 . \end{cases}$$

Simplifying, $\gamma_0 = \sigma^2_\varepsilon\left(\dfrac{1 - \cancel{\alpha_1^2} + \cancel{\alpha_1^2} + 2\alpha_1\beta_1 + \beta_1^2}{1-\alpha_1^2}\right) = \sigma^2_\varepsilon\left(\dfrac{1 + 2\alpha_1\beta_1 + \beta_1^2}{1-\alpha_1^2}\right)$

Simplifying, $\gamma_1 = \sigma^2_\varepsilon\left[(\alpha_1+\beta_1) + \dfrac{\alpha_1(\alpha_1+\beta_1)^2}{1-\alpha_1^2}\right] = \sigma^2_\varepsilon\left[\dfrac{(\alpha_1+\beta_1)(1-\alpha_1^2) + \alpha_1(\alpha_1+\beta_1)^2}{1-\alpha_1^2}\right]$

$$= \sigma^2_\varepsilon\left[\frac{\alpha_1 - \cancel{\alpha_1^3} + \beta_1 - \alpha_1^2\beta_1 + \cancel{\alpha_1^3} + 2\alpha_1^2\beta_1 + \alpha_1\beta_1^2}{1-\alpha_1^2}\right] = \sigma^2_\varepsilon\left[\frac{(1+\alpha_1\beta_1)(\alpha_1+\beta_1)}{1-\alpha_1^2}\right]$$

$$\Rightarrow \rho_1 = \frac{\cancel{\sigma^2_\varepsilon}\left[\dfrac{(1+\alpha_1\beta_1)(\alpha_1+\beta_1)}{\cancel{1-\alpha_1^2}}\right]}{\cancel{\sigma^2_\varepsilon}\left(\dfrac{1+2\alpha_1\beta_1+\beta_1^2}{\cancel{1-\alpha_1^2}}\right)} = \frac{(1+\alpha_1\beta_1)(\alpha_1+\beta_1)}{1+2\alpha_1\beta_1+\beta_1^2} .$$

etc.