Topic: Time series modelling

- This problem set deals with estimation of univariate time series models, using the Box-Jenkins ARIMA methodology.
- We use three datasets: (1) 177 quarterly observations on log U.S. GNP from 1947Q1 to 1991Q1, available as `gnp.txt` on the website. (2) 60 months of bond yield (interest rate) data from 1990M1 to 1994M12, available as `bond_yields.txt`. (3) 288 months of total France-U.S. scheduled air passenger traffic data from 1983M1 to 2006M12, available as `france_us_total_traffic_monthly.txt`.

Box-Jenkins methodology

- The central idea is to fit the 'best' univariate ARIMA (integrated ARMA) model to the time series of interest. There is no reason why an ARMA(p, q) model with small p and q should adequately describe a given time series. However, as a recipe for building short-term forecasting models, it provides a useful set of theoretical tools, and a workable empirical counterpart.
- The *Wold decomposition theorem* states that every covariance stationary stochastic process can be expressed as an ARMA(p, ∞) with white noise errors. Practically, we cannot estimate an ARMA(p, ∞), and a compromise is to base the estimation on a model with a finite number of ARMA terms and choose the best ARMA(p, q) *approximation* that fits the data, where p and q are generally small. Part of the model building process will therefore involve determination of the lag structure.

- **(step 1)** Wold's decomposition theorem applies to stationary stochastic processes, and so the data y_t must first be *transformed* to stationarity (if it is not already stationary). Essentially, this implies a sample autocorrelation function (SACF) that decays 'rapidly'. Usually, the transformation involves taking first differences Δy_t or second differences $\Delta^2 y_t$. Simply, this choice can be made by a visual assessment of the SACF, or by fitting an AR(1) with constant to the data and checking whether the AR parameter is 'close to' 1 (a 'unit root'). Rigorously, unit root tests must be used to decide whether a variable is stationary or nonstationary (not considered).

- **(step 2)** Next, we must *estimate the parameters* of a variety of (usually low-order) ARMA(p, q) models, by nonlinear least squares or maximum likelihood. The most parsimonious model (the best fit, subject to not having 'too many' parameters) is selected by use of an information criterion such as the *Schwarz Information Criterion* (SIC). It is also important to check the usual diagnostics (e.g. significance of variables), and also to compare the SACF and sample partial ACF (SPACF) of the data with those implied by the model (the ACF and PACF).

- **(step 3)** Once a model has been selected, the *model residuals* must be analyzed, to check whether they resemble white noise (using the *Ljung-Box Q-statistic*, which is based upon the ACF of the estimated residuals).

- **(step 4)** Once a parsimonious ARMA model has been specified and estimated, it can be used for *forecasting*, and the quality of the forecasts can be checked using formal tests (these tests are not considered here).

- **Ljung-Box Q-statistic**: This test uses the ACF ρ_k to test H_0 : there is no significant autocorrelation *up to and including* lag s, against the two-sided alternative H_1 : at least one autocorrelation is significant, up to and including lag s. The statistic is given by

$$Q = T(T+2)\sum_{k=1}^{s}\frac{\rho_k^2}{T-k} \sim \chi^2(s-h),$$

where T is the effective sample size, s is the maximum lag, ρ_k is either the ACF or the SACF, and $h = 0$ for raw data, and $h = p + q$ for ARMA(p, q) model residuals. Under H_0, all ρ_k will be close to 0, and so Q will be small.

- **Schwarz Information Criterion (SIC)**: How well does an estimated model fit the data? There is a trade-off between *improving model fit* by including an additional ARMA parameter ($p+q$ increases), against *parsimony* of the model (inclusion of too many coefficients can lead to over-fitting and poor forecasting performance). The SIC is asymptotically consistent for ARMA(p, q) models, i.e. given a 'large enough' sample and consideration of a set of p and q that *includes the true* p, q, the SIC will select the correct lag-order. The 'best' ARMA model is chosen by minimizing the SIC across various ARMA(p, q) models, e.g. $p, q = 0, 1, 2, 3, 4$.

Illustration of Box-Jenkins methodology using GNP data

1. Import the log GNP data into a dated (quarterly) workfile as y_t.

2. Examine the SACF and the SPACF of y_t. Note that the SACF decays slowly, while the SPACF is close to zero for lag greater than 1. This is strongly suggestive of an AR(1) with positive coefficient (the SACF is always positive): see the lecture notes.

3. Without transforming the data, fit an AR(1) model with constant to y_t, and examine the AR parameter. If this is 'close to' 1 (roughly, $|\widehat{\alpha_1}| > 0.95$), then the data may be considered to be nonstationary, and will need to be transformed. Also examine the roots of the inverse polynomial associated with the AR lag polynomial. What do you notice? Then, compare the data and model ACF/PACF (the SACF and model

ACF, and the SPACF and model PACF). What do you see? Finally, check whether the estimated residuals are white noise.

4. Again without transforming the data, fit an AR(2) model with constant to y_t. As for the AR(1), assess the quality of the model, and check for nonstationarity of y_t.

5. Now transform y_t by taking the first-difference, i.e. $\Delta y_t := y_t - y_{t-1}$. Plot the series (which can be interpreted as the quarterly growth rate of GNP), and fit an AR(1) model with constant to Δy_t. Assess the quality of the model fit.

6. Fit an MA(2) model with constant to Δy_t. Assess the quality of the model fit. Compare the SIC from the MA(2) model to the SIC from the AR(1) model. What do you see?

7. Fit an ARMA(2,2) model with constant to Δy_t. Assess the quality of the model fit, and in particular the data/model ACF and PACF. Compare the SIC from the ARMA(2,2) model to the SIC from the ARMA(p, q) models with $p, q = 0, 1, 2$ (all combinations). What do you see? **Note that minimizing SIC is less important than finding a parsimonious model with good properties, and that captures the ACF and PACF dependence structure**...

8. Finally, note that an ARMA(p, q) model fitted to transformed (differenced) data is known as an ARIMA (integrated ARMA) model. For instance, an ARMA(2,2) fitted to Δy_t is an ARIMA(2,1,2) model, where the second parameter in brackets refers to the degree of differencing of the data (to transform it to stationarity). Likewise, an ARMA(p, q) model fitted to $\Delta^d y_t$ is an ARIMA(p, d, q) model.

An ARIMA model for interest rates

1. (There are no figures to help you in this section: the method is as above)

2. Import the bond yield data into a dated (monthly) workfile as y_t. Plot the series, and examine the SACF and SPACF. Does the series y_t appear to be stationary? What ARMA model do the SACF and SPACF suggest might be appropriate for y_t? Do you notice anything unusual in the SACF?!

3. Compute (manually) the Ljung-Box Q statistic (this is also reported by EViews) for the autocorrelations of y_t, and test that there is no jointly significant autocorrelation up to and including (i) lag $s = 1$, (ii) lag $s = 2$, and (iii) lag $s = 3$, at the 95% level of significance. Based on these results, is the series y_t a white noise process $y_t \sim \text{ARMA}(0, 0)$?

4. Fit an AR(1) model with constant to y_t, with White's standard errors, and interpret the estimation output. Check whether the estimated model (and hence the data) seems to be stationary, by examining the roots of the inverse polynomial. (Note that you cannot rigorously test whether the data is nonstationary, i.e. root $= 1$ against stationary, i.e. $|\text{root}| < 1$ using a standard t statistic, since this statistic will not have its usual asymptotic normal distribution if the data is actually nonstationary).

5. Assuming that y_t is *not* stationary, create and plot the series Δy_t. Fit an AR(1) model with constant to the Δy_t. What do you notice about the regression output?

6. Fit ARMA(p, q) models to Δy_t, for all combinations of $p, q = 0, 1, 2, 3$. You do not need to use robust standard errors. Based on the SIC, which ARMA model provides the best fit to the transformed data?

7. For the best model from part 6 above, re-estimate the model over the sample period

1990M1 to 1994M5, using White's standard errors, and specifying the dependent variable in the equation as 'd(y)' (this syntax is needed for forecasting applications, if you wish to be able to forecast y_t (which is usually the case) rather than Δy_t). Carefully interpret the regression output. Plot and interpret the actual and fitted series, the roots of the inverse polynomials, the residual SACF and SPACF and Ljung-Box Q-statistics, and the data/model ACF/SACF and PACF/SPACF. Check for normality of the estimated residuals.

8. For the model estimated in part 7 above, perform (i) a *dynamic* forecast of the series y_t over the period 1994M6 to 1994M12 (save the forecast as 'yf' and the forecast standard error as 'yf_se'), and (ii) a *static* forecast of the series y_t over the period 1994M6 to 1994M12 (save the forecast as 'yf2'). Plot the original series y_t against 'yf', 'yf+2yf_se', 'yf−2yf_se' and 'yf2', and compare. (Note that a static forecast computes a sequence of one-step-ahead forecasts of the dependent variable, replacing any right-hand-side lagged dependent variables with their actual values, or with actual estimated values for MA terms. A dynamic forecast, on the other hand, will replace right-hand-side lagged dependent variables with their *previously forecasted* values). The plot for the best model is given below.

An ARIMA model for passenger traffic (the final step!)

1. (There are no figures to help you in this section. Import the passenger traffic data over 1983M1 to 2006M12 as y_t. Using all appropriate techniques, construct both dynamic and static forecasts for the out-of-sample period 2006M1 to 2006M12. **Hint**: before attempting Box-Jenkins modelling, first *remove* any strong linear time trends and/or deterministic seasonality from y_t, e.g. of the form $\theta_1 t$ and/or $\theta_2 \cos(2\pi/M)(t-\theta_3)$, where M determines the period of the seasonality, and is to be fixed (chosen, not

estimated) in advance. Also, account for any structural breaks that may be relevant. Interpret all of your results carefully. How well do your forecasts track the actual 2006 total passenger numbers? (**Advanced**: you may also consider seasonal ARMA models: see the EViews manual for details!)

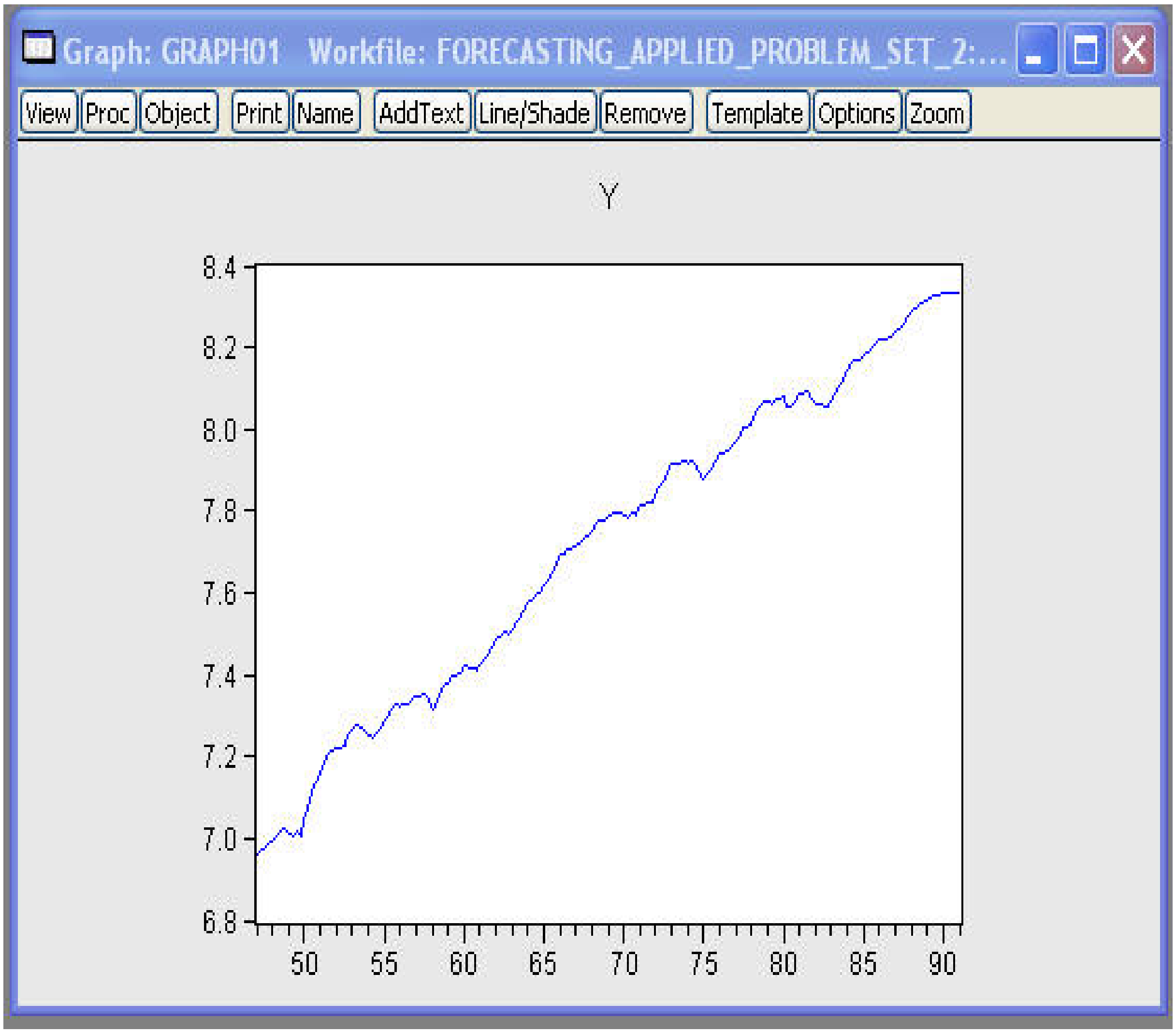
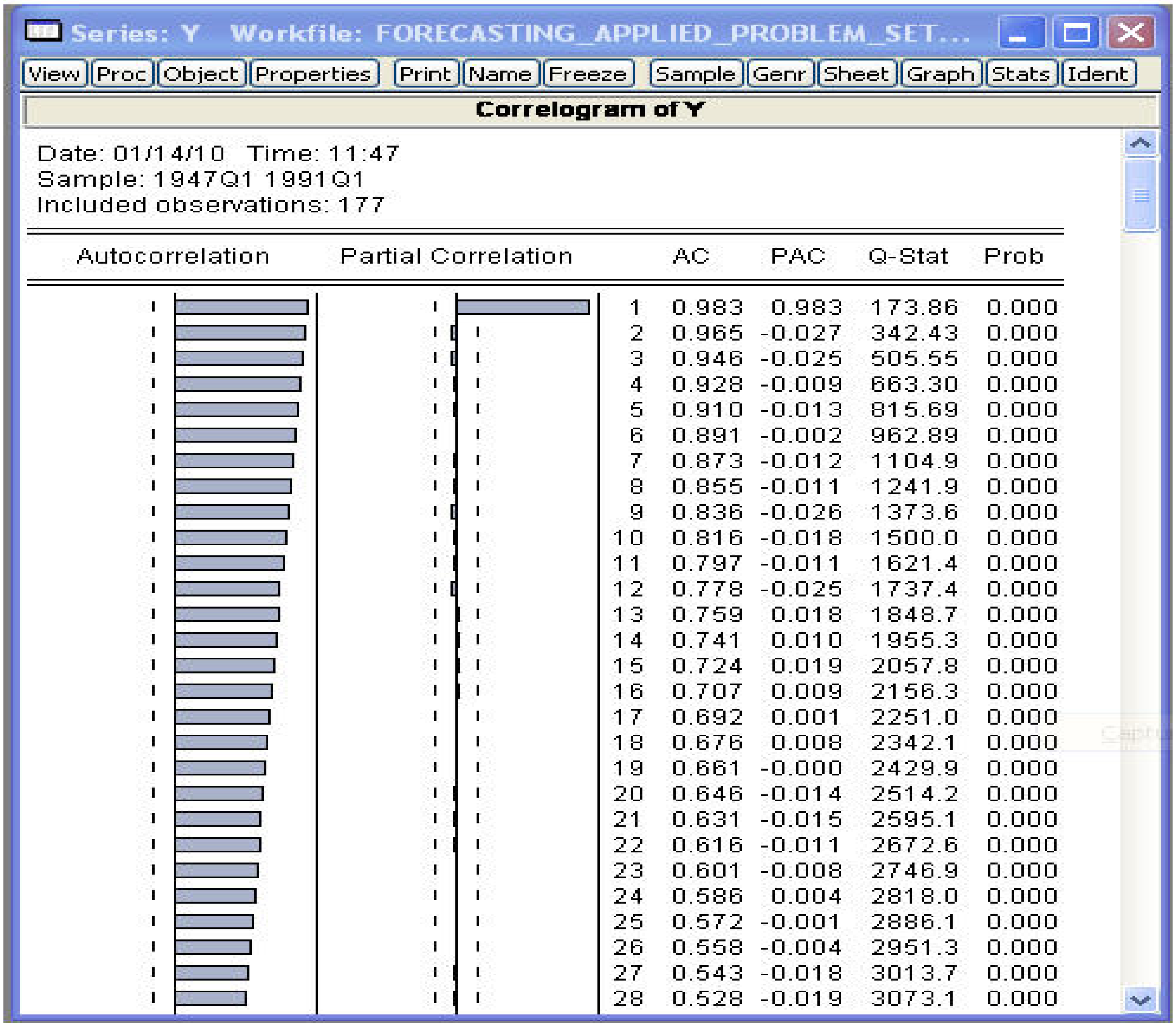


Figure 1: Log of quarterly U.S. GNP, y_t, from 1947Q1 to 1991Q1.



Series: Y Workfile: FORECASTING_APPLIED_PROBLEM_SET...

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Correlogram of Y

Date: 01/14/10 Time: 11:47
Sample: 1947Q1 1991Q1
Included observations: 177

Lag	AC	PAC	Q-Stat	Prob
1	0.983	0.983	173.86	0.000
2	0.965	-0.027	342.43	0.000
3	0.946	-0.025	505.55	0.000
4	0.928	-0.009	663.30	0.000
5	0.910	-0.013	815.69	0.000
6	0.891	-0.002	962.89	0.000
7	0.873	-0.012	1104.9	0.000
8	0.855	-0.011	1241.9	0.000
9	0.836	-0.026	1373.6	0.000
10	0.816	-0.018	1500.0	0.000
11	0.797	-0.011	1621.4	0.000
12	0.778	-0.025	1737.4	0.000
13	0.759	0.018	1848.7	0.000
14	0.741	0.010	1955.3	0.000
15	0.724	0.019	2057.8	0.000
16	0.707	0.009	2156.3	0.000
17	0.692	0.001	2251.0	0.000
18	0.676	0.008	2342.1	0.000
19	0.661	-0.000	2429.9	0.000
20	0.646	-0.014	2514.2	0.000
21	0.631	-0.015	2595.1	0.000
22	0.616	-0.011	2672.6	0.000
23	0.601	-0.008	2746.9	0.000
24	0.586	0.004	2818.0	0.000
25	0.572	-0.001	2886.1	0.000
26	0.558	-0.004	2951.3	0.000
27	0.543	-0.018	3013.7	0.000
28	0.528	-0.019	3073.1	0.000

Figure 2: SACF and SPACF of y_t, which suggests a low-order AR process, e.g. an AR(1).

Equation: EQ01 Workfile: FORECASTING_APPLIED_...

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Dependent Variable: Y
Method: Least Squares
Date: 01/14/10 Time: 11:53
Sample (adjusted): 1947Q2 1991Q1
Included observations: 176 after adjustments
Convergence achieved after 4 iterations

	Coefficient	Std. Error	t-Statistic	Prob.
C	9.762822	1.122667	8.696101	0.0000
AR(1)	0.996230	0.002024	492.3002	0.0000

R-squared	0.999283	Mean dependent var	7.717105
Adjusted R-squared	0.999278	S.D. dependent var	0.396573
S.E. of regression	0.010653	Akaike info criterion	-6.234724
Sum squared resid	0.019745	Schwarz criterion	-6.198696
Log likelihood	550.6557	Hannan-Quinn criter.	-6.220111
F-statistic	242359.5	Durbin-Watson stat	1.256012
Prob(F-statistic)	0.000000		

Inverted AR Roots	1.00

Figure 3: AR(1) regression $y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$ for log U.S. GNP, with very high R^2, significant coefficients, a low Durbin-Watson statistic, and a near unit root ($\widehat{\alpha_1} \approx 1$).

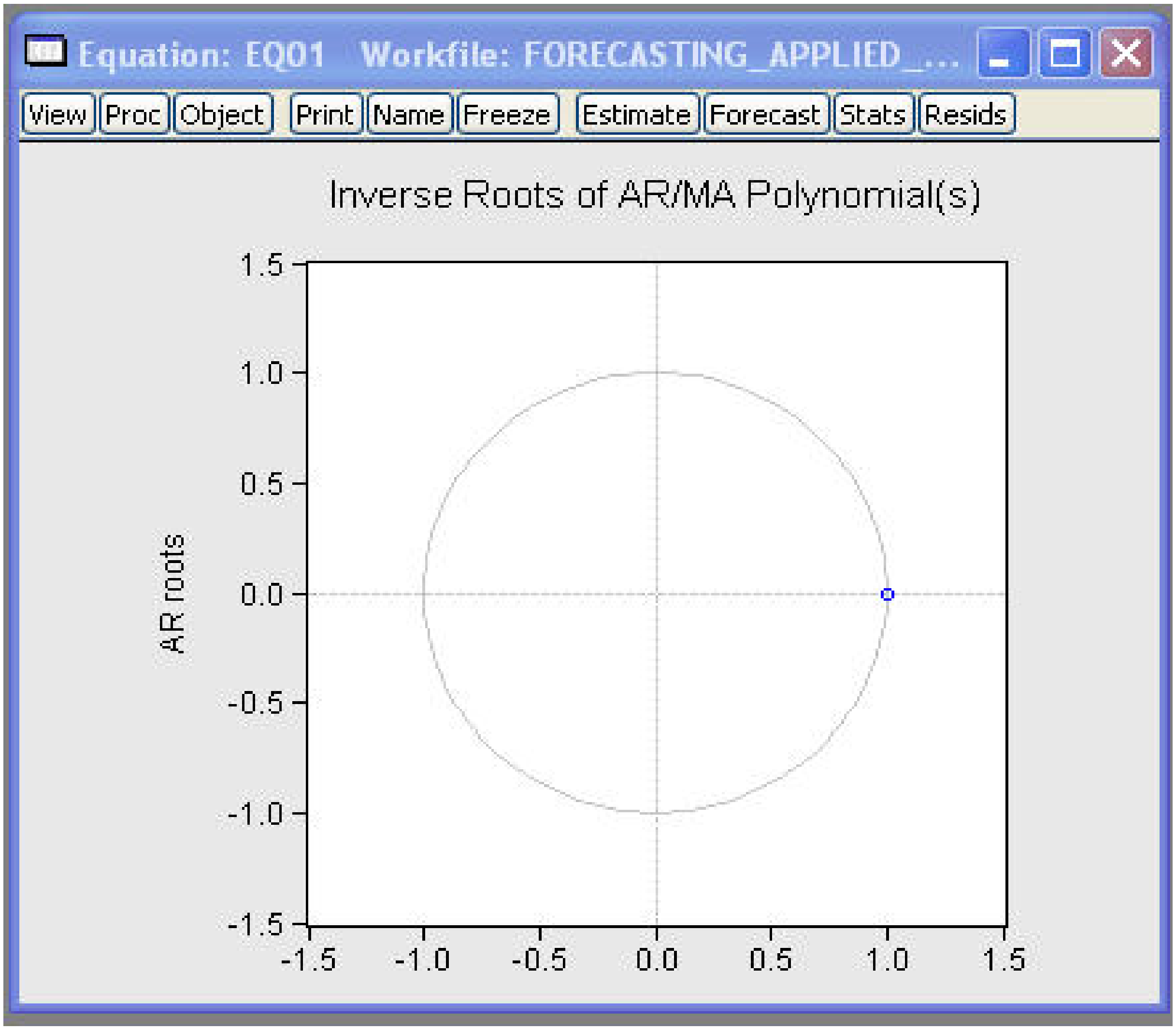


Figure 4: Root of AR(1) inverse polynomial $\alpha^{\star}(\lambda)$, indicating nonstationarity.

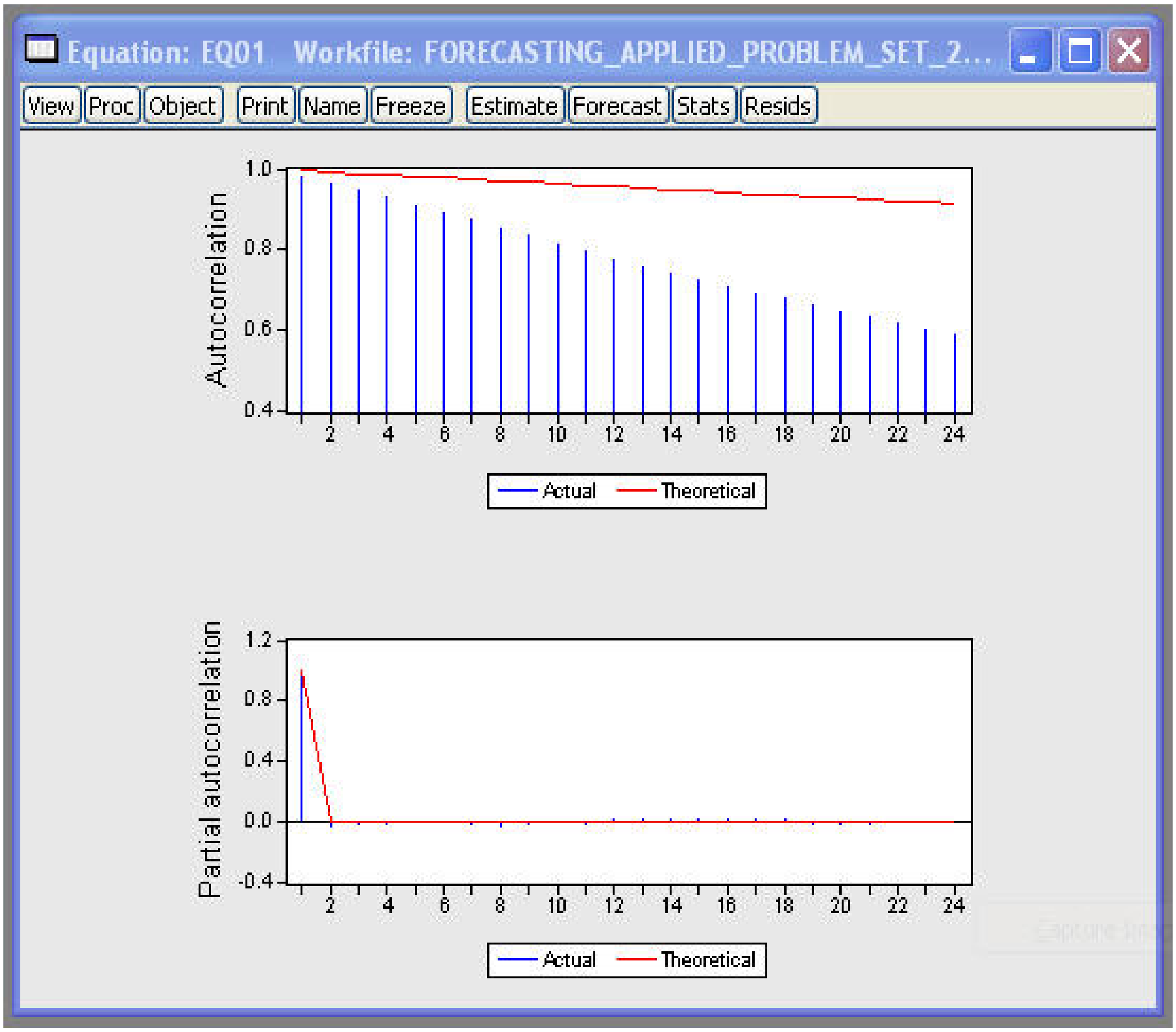
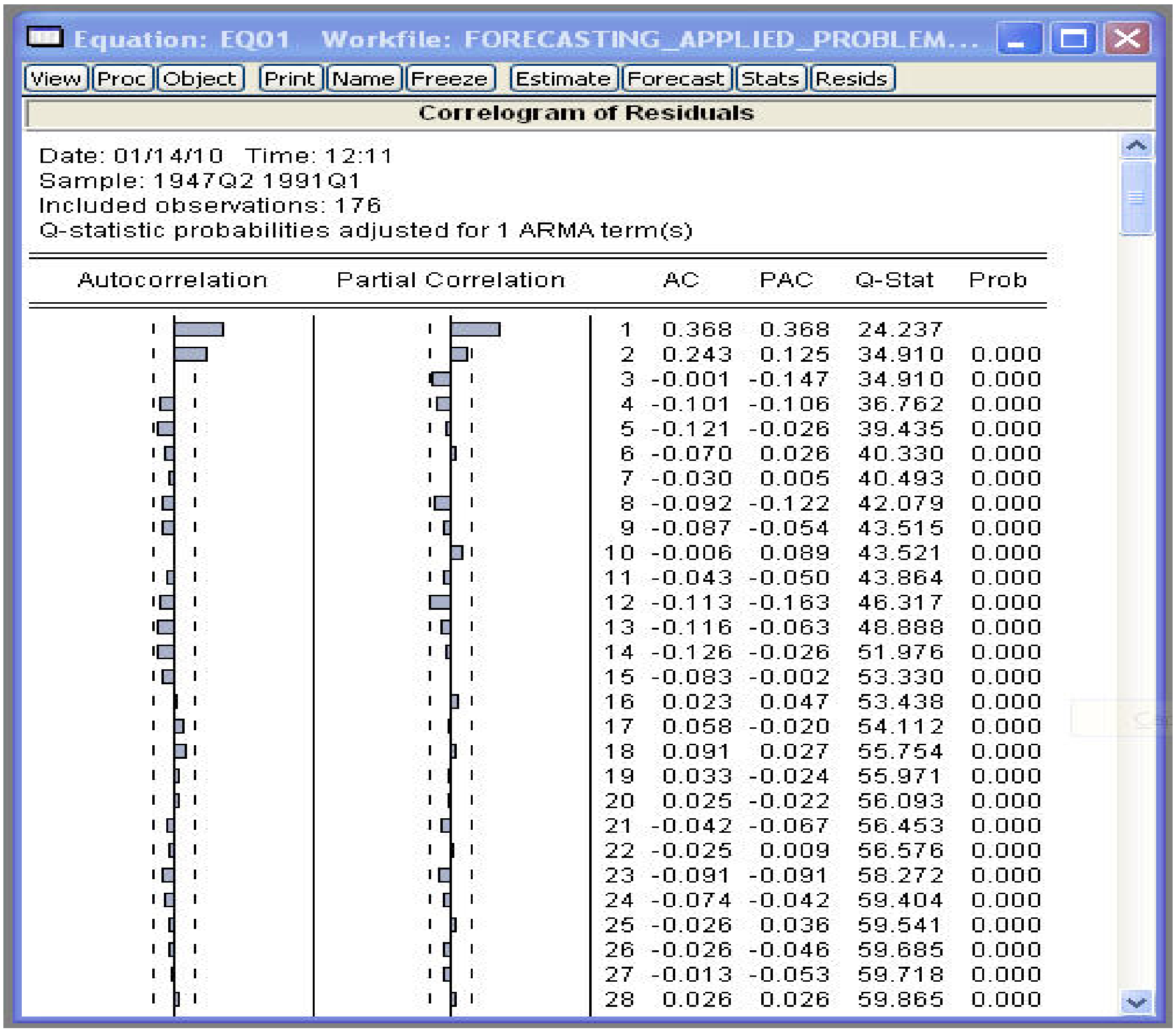


Figure 5: The AR(1) model ACF/data SACF and model PACF/data SPACF, which show that autocorrelation decays more rapidly in the data than the model would suggest. It is likely that the near nonstationarity of y_t is obscuring the underlying dependence in the data.



Equation: EQ01 Workfile: FORECASTING_APPLIED_PROBLEM...

View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Correlogram of Residuals

Date: 01/14/10 Time: 12:11
Sample: 1947Q2 1991Q1
Included observations: 176
Q-statistic probabilities adjusted for 1 ARMA term(s)

Lag	AC	PAC	Q-Stat	Prob
1	0.368	0.368	24.237	
2	0.243	0.125	34.910	0.000
3	-0.001	-0.147	34.910	0.000
4	-0.101	-0.106	36.762	0.000
5	-0.121	-0.026	39.435	0.000
6	-0.070	0.026	40.330	0.000
7	-0.030	0.005	40.493	0.000
8	-0.092	-0.122	42.079	0.000
9	-0.087	-0.054	43.515	0.000
10	-0.006	0.089	43.521	0.000
11	-0.043	-0.050	43.864	0.000
12	-0.113	-0.163	46.317	0.000
13	-0.116	-0.063	48.888	0.000
14	-0.126	-0.026	51.976	0.000
15	-0.083	-0.002	53.330	0.000
16	0.023	0.047	53.438	0.000
17	0.058	-0.020	54.112	0.000
18	0.091	0.027	55.754	0.000
19	0.033	-0.024	55.971	0.000
20	0.025	-0.022	56.093	0.000
21	-0.042	-0.067	56.453	0.000
22	-0.025	0.009	56.576	0.000
23	-0.091	-0.091	58.272	0.000
24	-0.074	-0.042	59.404	0.000
25	-0.026	0.036	59.541	0.000
26	-0.026	-0.046	59.685	0.000
27	-0.013	-0.053	59.718	0.000
28	0.026	0.026	59.865	0.000

Figure 6: ACF and PACF and Q-statistics for estimated residuals from the AR(1) model fitted to y_t, indicating that the residuals are not white noise.

Equation: EQ02 Workfile: FORECASTING_APPLIED_PROBLEM...

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Dependent Variable: Y
Method: Least Squares
Date: 01/14/10 Time: 12:44
Sample (adjusted): 1947Q3 1991Q1
Included observations: 175 after adjustments
Convergence achieved after 4 iterations

	Coefficient	Std. Error	t-Statistic	Prob.
C	9.413386	1.168754	8.054208	0.0000
AR(1)	1.367027	0.071069	19.23511	0.0000
AR(2)	-0.369884	0.070846	-5.220928	0.0000

R-squared	0.999368	Mean dependent var	7.721380
Adjusted R-squared	0.999361	S.D. dependent var	0.393623
S.E. of regression	0.009950	Akaike info criterion	-6.365508
Sum squared resid	0.017028	Schwarz criterion	-6.311254
Log likelihood	559.9819	Hannan-Quinn criter.	-6.343501
F-statistic	136071.3	Durbin-Watson stat	2.091968
Prob(F-statistic)	0.000000		

Inverted AR Roots	1.00	.37

Figure 7: AR(2) regression $y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \varepsilon_t$ for log U.S. GNP, with very high R^2, significant coefficients, and a 'good' Durbin-Watson statistic. Has the nonstationary behaviour disappeared? (See the roots of the inverse polynomial below).

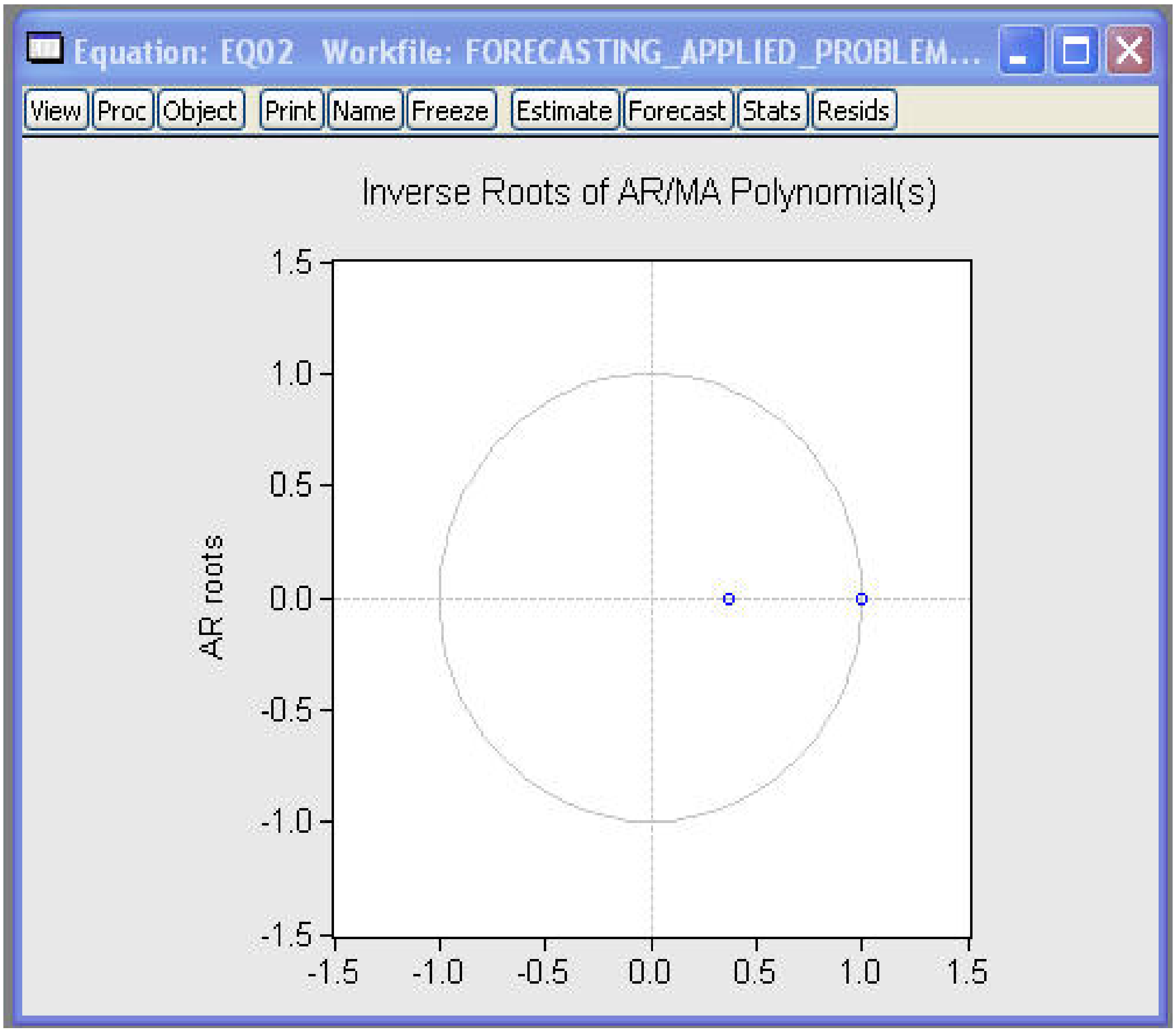


Figure 8: Roots of AR(2) inverse polynomial $\alpha^\star(\lambda)$, indicating that the nonstationary behaviour is still present! (While one root is less than 1 in absolute value, the other is very close to 1).

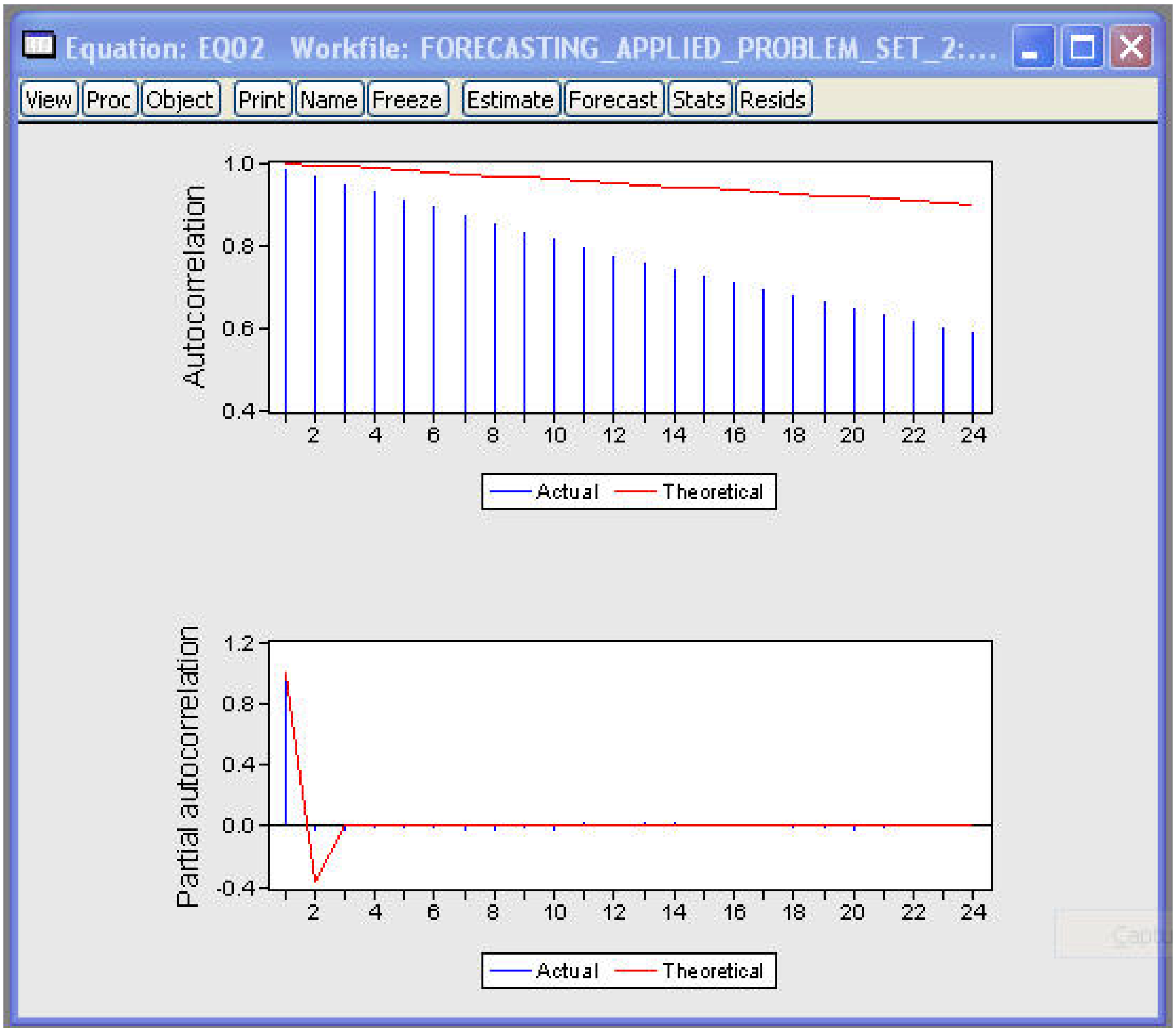
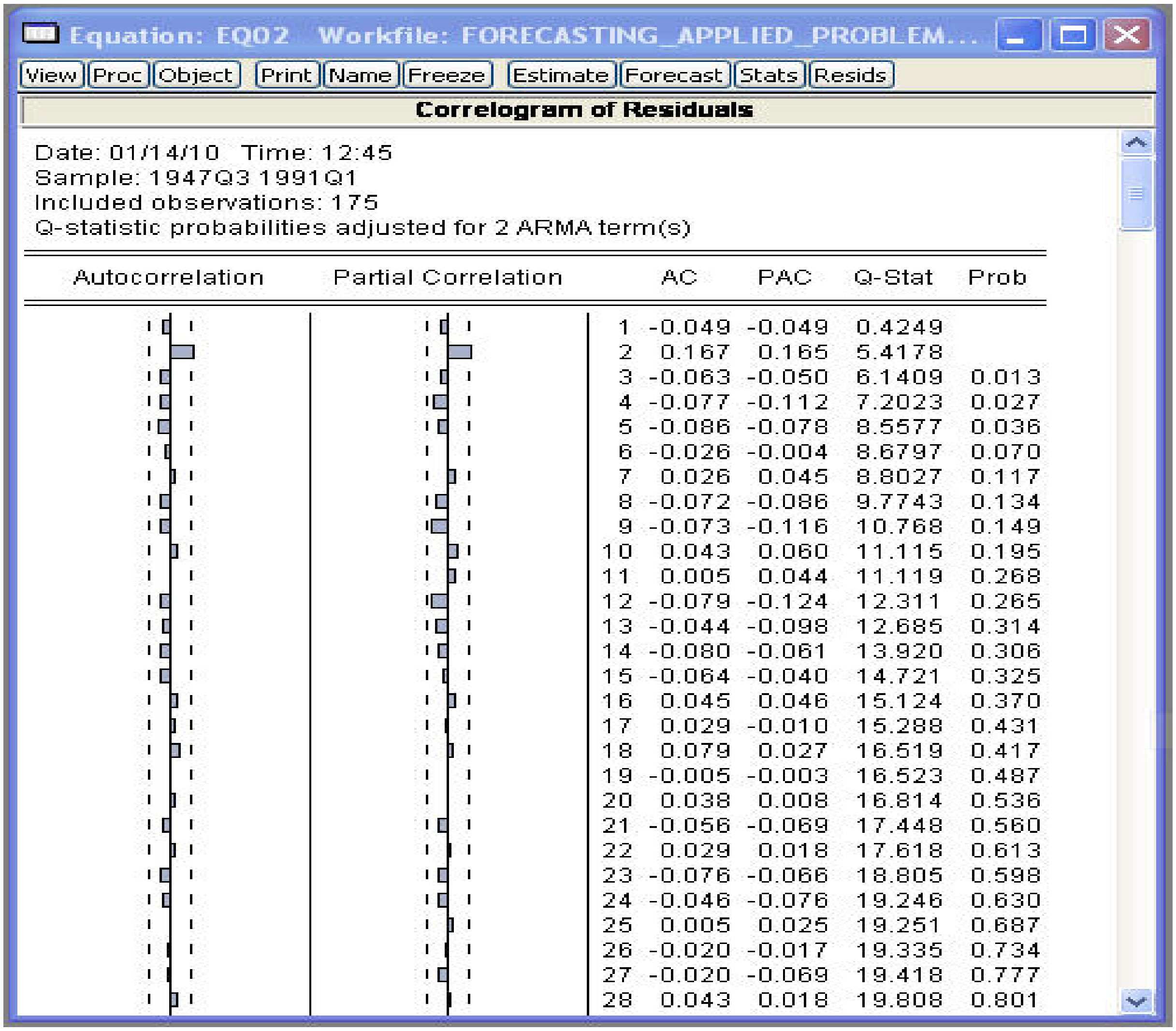


Figure 9: The AR(2) model ACF/data SACF and model PACF/data SPACF, which show that autocorrelation still decays more rapidly in the data than the model would suggest.



Equation: EQ02 Workfile: FORECASTING_APPLIED_PROBLEM...

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Correlogram of Residuals

Date: 01/14/10 Time: 12:45
Sample: 1947Q3 1991Q1
Included observations: 175
Q-statistic probabilities adjusted for 2 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	-0.049	-0.049	0.4249	
2	0.167	0.165	5.4178	
3	-0.063	-0.050	6.1409	0.013
4	-0.077	-0.112	7.2023	0.027
5	-0.086	-0.078	8.5577	0.036
6	-0.026	-0.004	8.6797	0.070
7	0.026	0.045	8.8027	0.117
8	-0.072	-0.086	9.7743	0.134
9	-0.073	-0.116	10.768	0.149
10	0.043	0.060	11.115	0.195
11	0.005	0.044	11.119	0.268
12	-0.079	-0.124	12.311	0.265
13	-0.044	-0.098	12.685	0.314
14	-0.080	-0.061	13.920	0.306
15	-0.064	-0.040	14.721	0.325
16	0.045	0.046	15.124	0.370
17	0.029	-0.010	15.288	0.431
18	0.079	0.027	16.519	0.417
19	-0.005	-0.003	16.523	0.487
20	0.038	0.008	16.814	0.536
21	-0.056	-0.069	17.448	0.560
22	0.029	0.018	17.618	0.613
23	-0.076	-0.066	18.805	0.598
24	-0.046	-0.076	19.246	0.630
25	0.005	0.025	19.251	0.687
26	-0.020	-0.017	19.335	0.734
27	-0.020	-0.069	19.418	0.777
28	0.043	0.018	19.808	0.801

Figure 10: ACF and PACF and Q-statistics for estimated residuals from the AR(2) model fitted to y_t, indicating that the residuals are approximately white noise. However, the near nonstationarity of y_t means that the model is still not satisfactory.

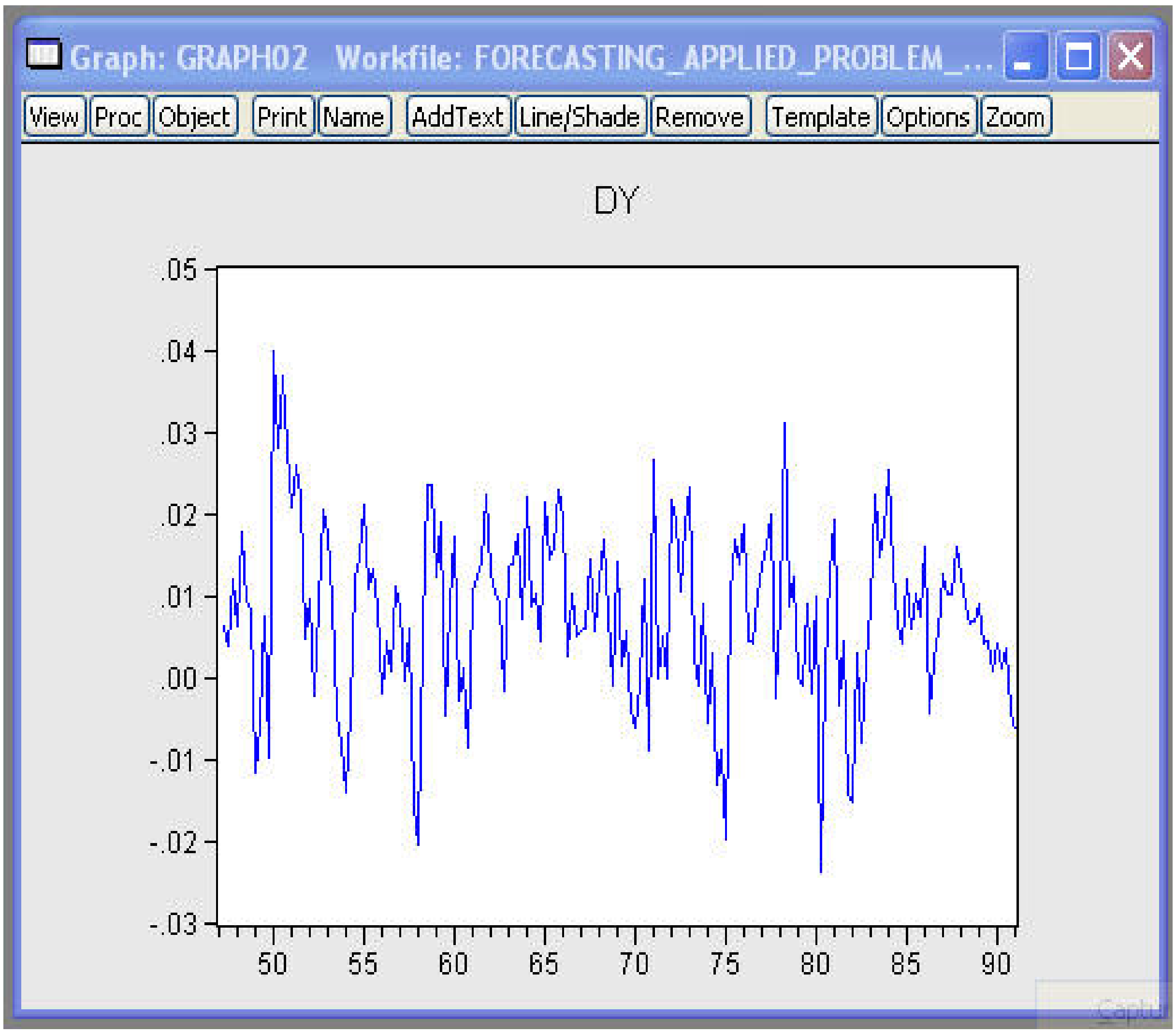

Figure 11: Quarterly U.S. GNP growth, Δy_t, from 1947Q2 to 1991Q1.

Equation: EQ03 Workfile: FORECASTING_APPLIED_PROBLEM...

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Dependent Variable: DY
Method: Least Squares
Date: 01/14/10 Time: 13:13
Sample (adjusted): 1947Q3 1991Q1
Included observations: 175 after adjustments
Convergence achieved after 3 iterations

	Coefficient	Std. Error	t-Statistic	Prob.
C	0.007704	0.001219	6.320839	0.0000
AR(1)	0.380721	0.070718	5.383637	0.0000

R-squared	0.143495	Mean dependent var	0.007749
Adjusted R-squared	0.138544	S.D. dependent var	0.010758
S.E. of regression	0.009985	Akaike info criterion	-6.364143
Sum squared resid	0.017247	Schwarz criterion	-6.327974
Log likelihood	558.8625	Hannan-Quinn criter.	-6.349472
F-statistic	28.98355	Durbin-Watson stat	2.095356
Prob(F-statistic)	0.000000		

Inverted AR Roots	.38

Figure 12: AR(1) regression $\Delta y_t = \alpha_0 + \alpha_1 \Delta y_{t-1} + \varepsilon_t$, with low R^2 (is this unexpected?!), significant coefficients, 'good' Durbin-Watson statistic, and stationary characteristics. It is unusual to go beyond Δ^2 in applied work.

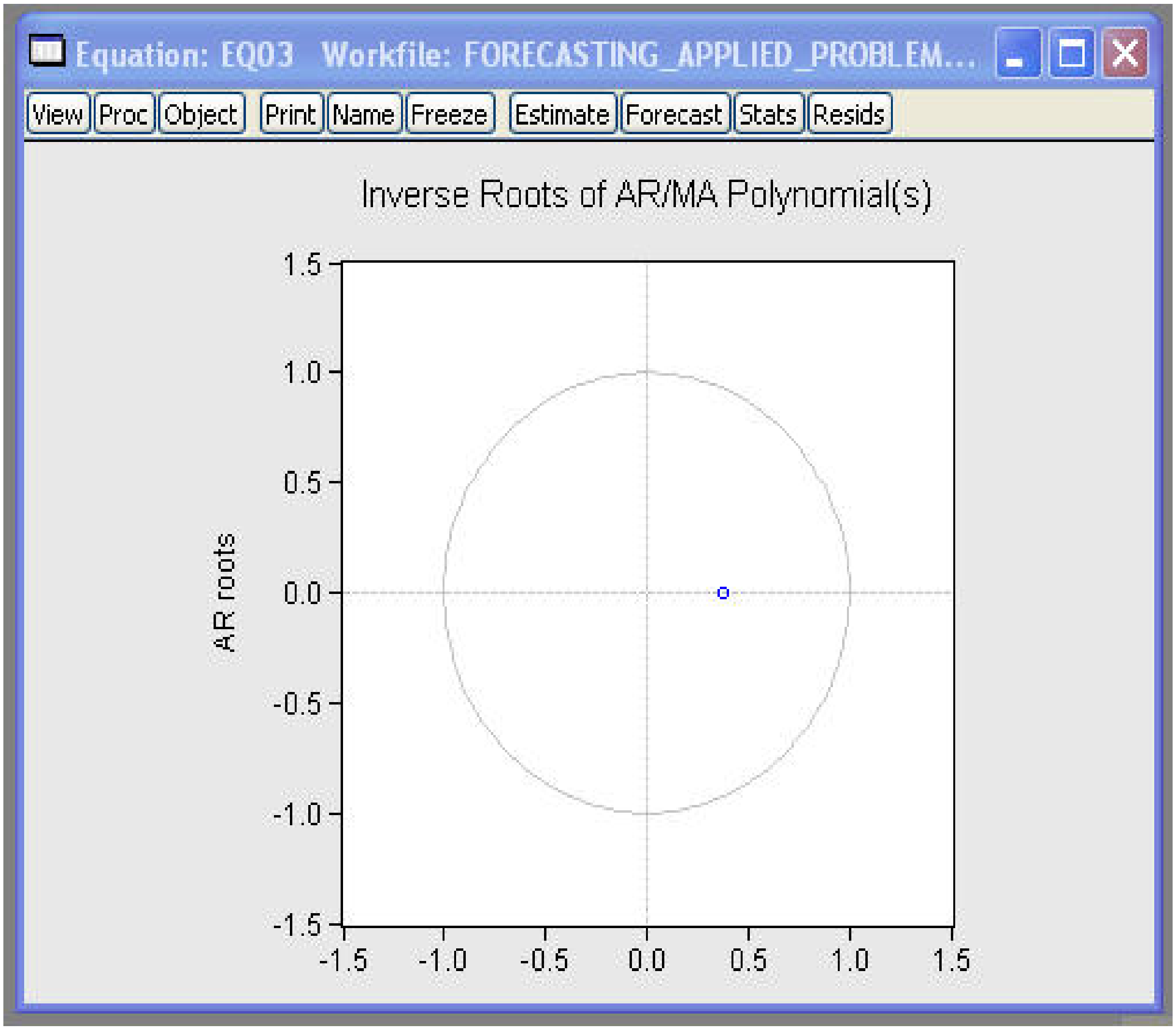


Figure 13: Roots of AR(1) inverse polynomial $\alpha^{\star}(\lambda)$, indicating that the model is stationary.

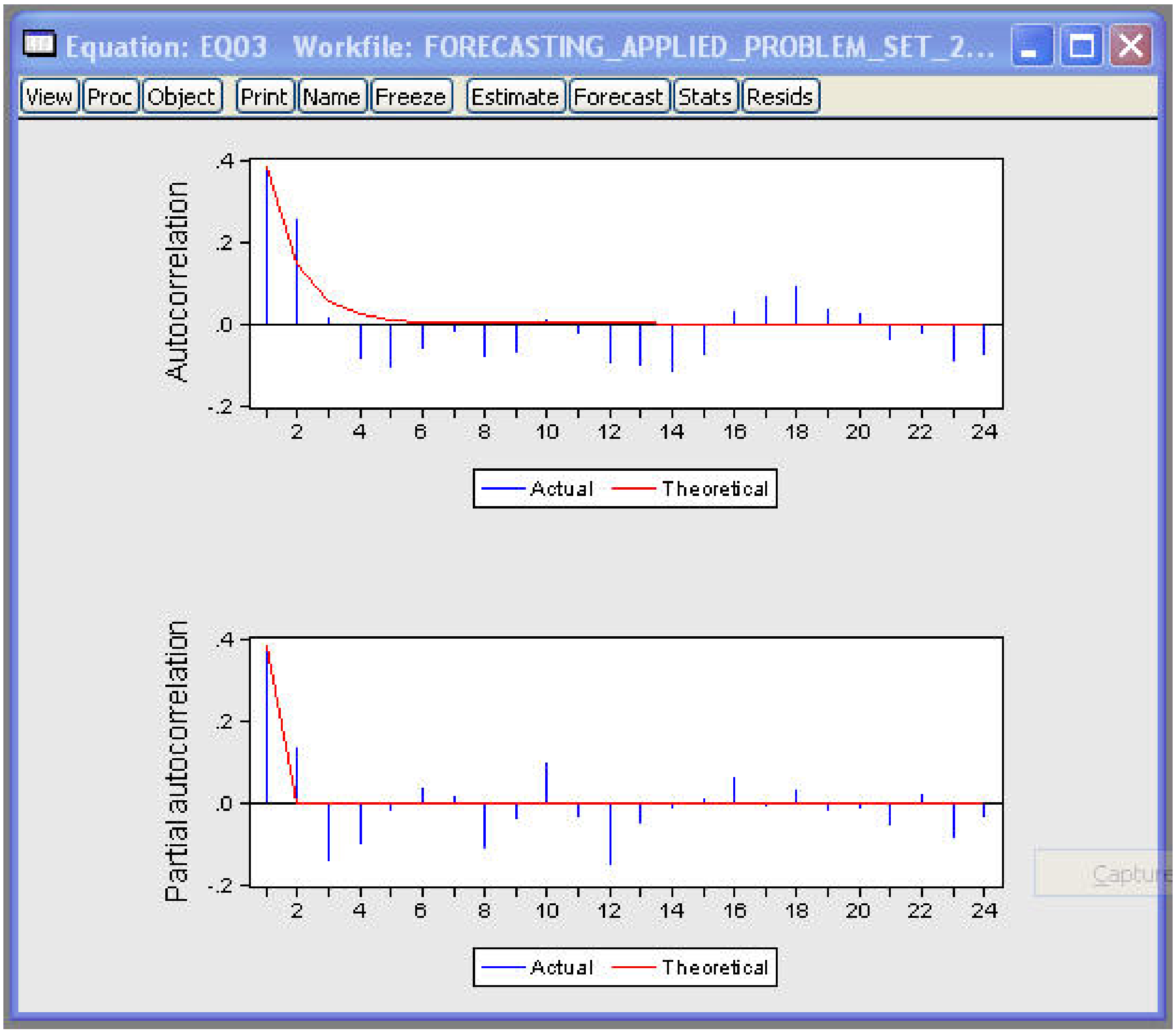
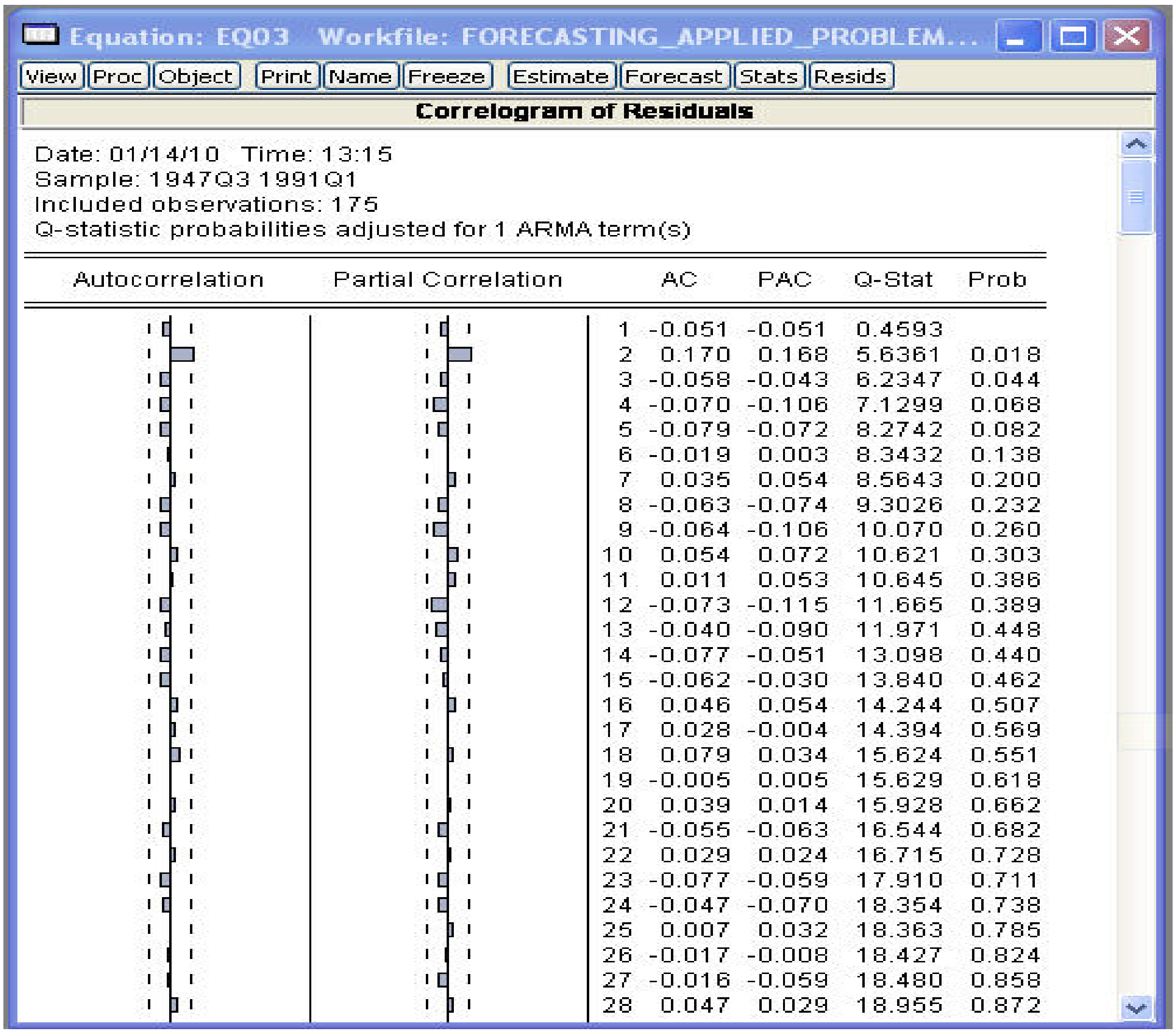


Figure 14: The AR(1) model ACF/data SACF and model PACF/data SPACF, which show that the data autocorrelation decays much more rapidly than before transformation. However, the model-implied ACF and PACF do not successfully capture the dependence in the data.


Equation: EQ03 Workfile: FORECASTING_APPLIED_PROBLEM...

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Correlogram of Residuals

Date: 01/14/10 Time: 13:15
Sample: 1947Q3 1991Q1
Included observations: 175
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	-0.051	-0.051	0.4593	
2	0.170	0.168	5.6361	0.018
3	-0.058	-0.043	6.2347	0.044
4	-0.070	-0.106	7.1299	0.068
5	-0.079	-0.072	8.2742	0.082
6	-0.019	0.003	8.3432	0.138
7	0.035	0.054	8.5643	0.200
8	-0.063	-0.074	9.3026	0.232
9	-0.064	-0.106	10.070	0.260
10	0.054	0.072	10.621	0.303
11	0.011	0.053	10.645	0.386
12	-0.073	-0.115	11.665	0.389
13	-0.040	-0.090	11.971	0.448
14	-0.077	-0.051	13.098	0.440
15	-0.062	-0.030	13.840	0.462
16	0.046	0.054	14.244	0.507
17	0.028	-0.004	14.394	0.569
18	0.079	0.034	15.624	0.551
19	-0.005	0.005	15.629	0.618
20	0.039	0.014	15.928	0.662
21	-0.055	-0.063	16.544	0.682
22	0.029	0.024	16.715	0.728
23	-0.077	-0.059	17.910	0.711
24	-0.047	-0.070	18.354	0.738
25	0.007	0.032	18.363	0.785
26	-0.017	-0.008	18.427	0.824
27	-0.016	-0.059	18.480	0.858
28	0.047	0.029	18.955	0.872

Figure 15: ACF and PACF and Q-statistics for estimated residuals from the AR(1) model fitted to Δy_t, indicating that the residuals are approximately white noise. However, the poor fit of the oscillatory decay in the ACF/PACF suggests that a better model can be found.

Equation: EQ04 Workfile: FORECASTING_APPLIED_PROBLEM_...

View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Dependent Variable: DY
Method: Least Squares
Date: 01/14/10 Time: 13:25
Sample (adjusted): 1947Q2 1991Q1
Included observations: 176 after adjustments
Convergence achieved after 9 iterations
MA Backcast: 1946Q4 1947Q1

	Coefficient	Std. Error	t-Statistic	Prob.
C	0.007688	0.001174	6.547759	0.0000
MA(1)	0.313387	0.073318	4.274373	0.0000
MA(2)	0.274145	0.073443	3.732739	0.0003

R-squared	0.169512	Mean dependent var	0.007741
Adjusted R-squared	0.159911	S.D. dependent var	0.010728
S.E. of regression	0.009832	Akaike info criterion	-6.389353
Sum squared resid	0.016725	Schwarz criterion	-6.335311
Log likelihood	565.2631	Hannan-Quinn criter.	-6.367434
F-statistic	17.65559	Durbin-Watson stat	1.941726
Prob(F-statistic)	0.000000		

Inverted MA Roots	-.16+.50i	-.16-.50i

Figure 16: MA(2) regression $\Delta y_t = \alpha_0 + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2}$, with low R^2, significant coefficients, 'good' Durbin-Watson statistic, and stationary characteristics (of course!).

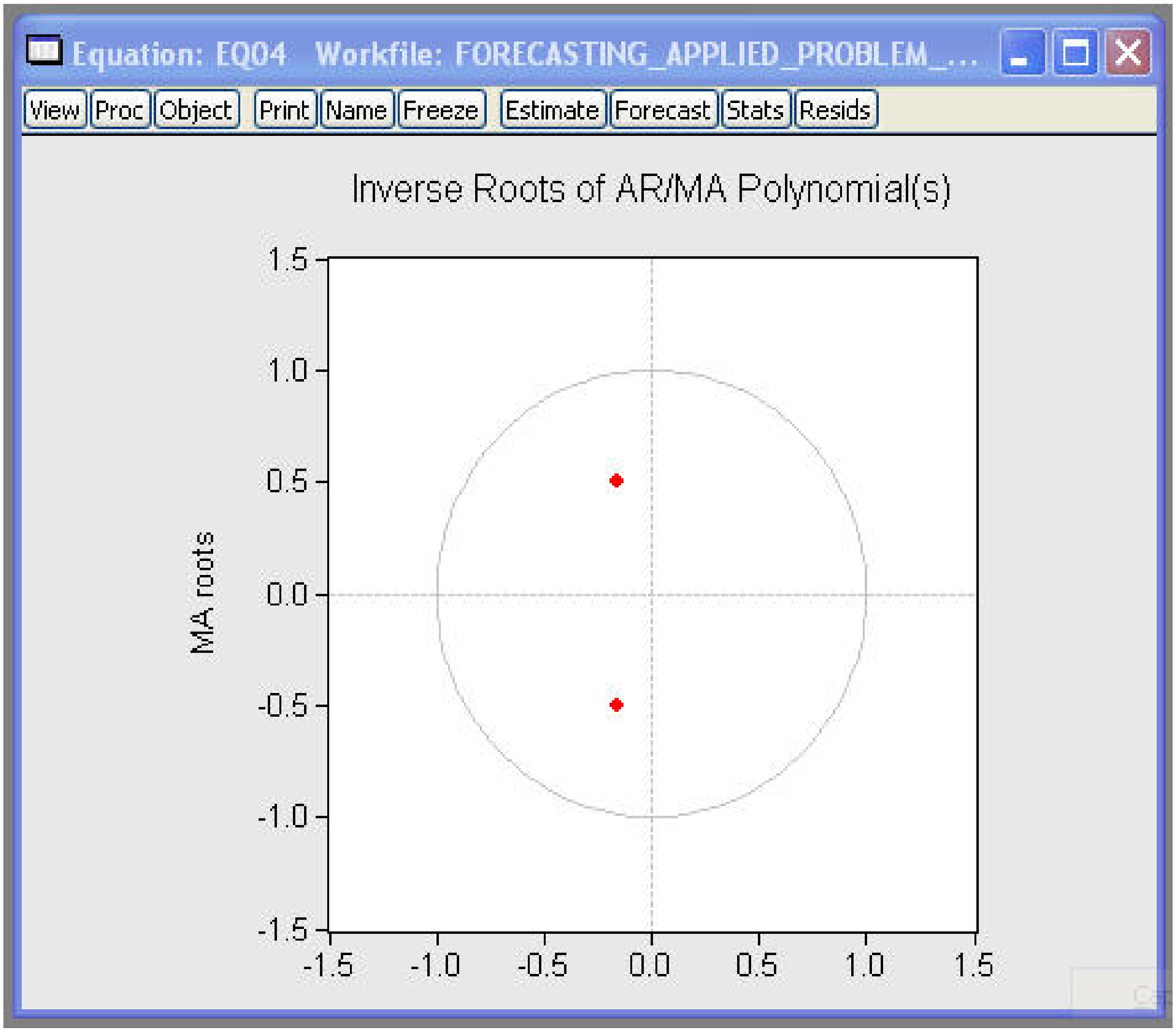


Figure 17: Roots of MA(2) inverse polynomial $\beta^{\star}(\lambda)$, indicating that the model is *invertible*. Note that all MA models are stationary, and that invertibility has no impact on stationarity (however, a non-invertible model often indicates numerical problems in the estimation, and should be checked carefully).

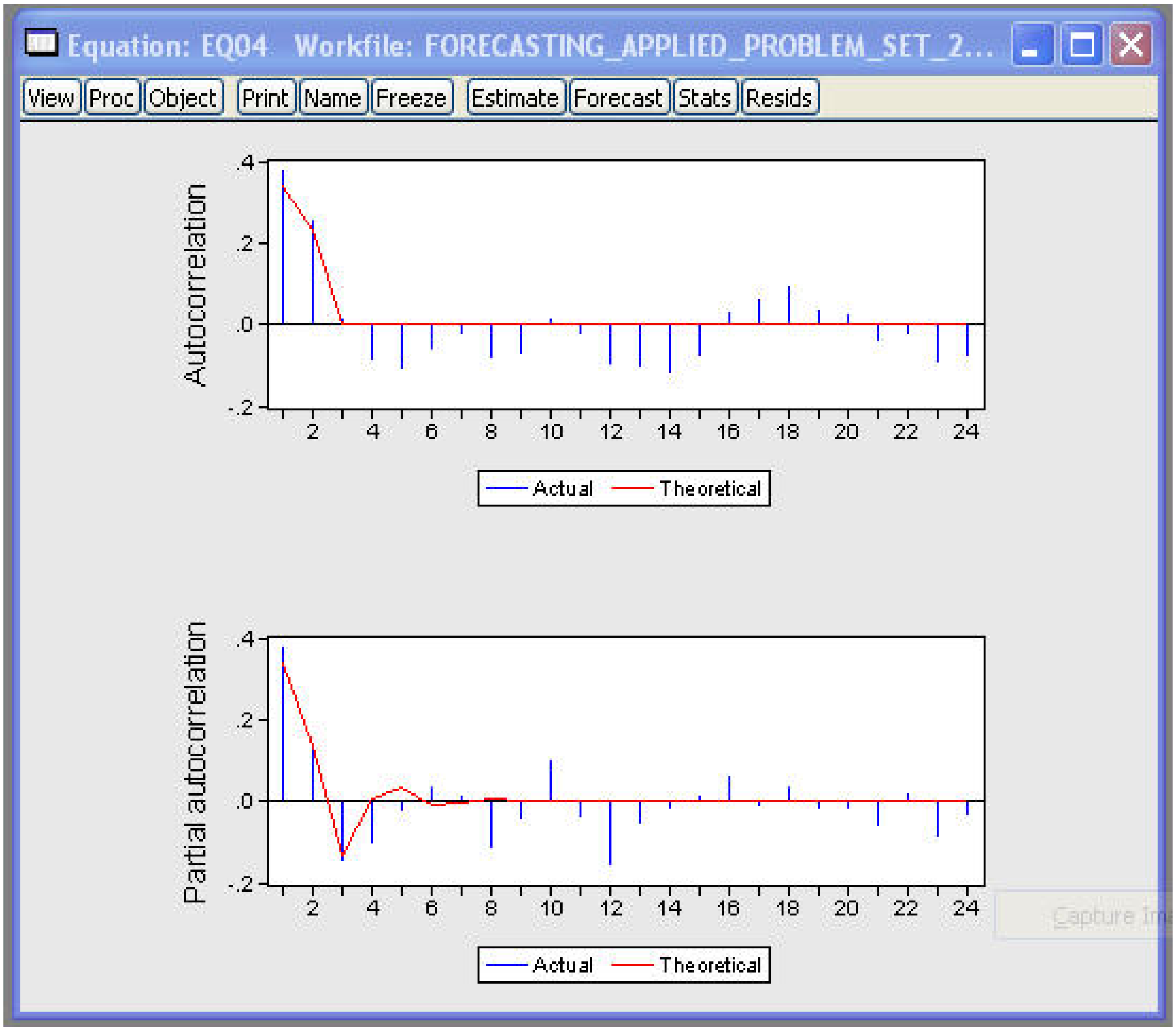
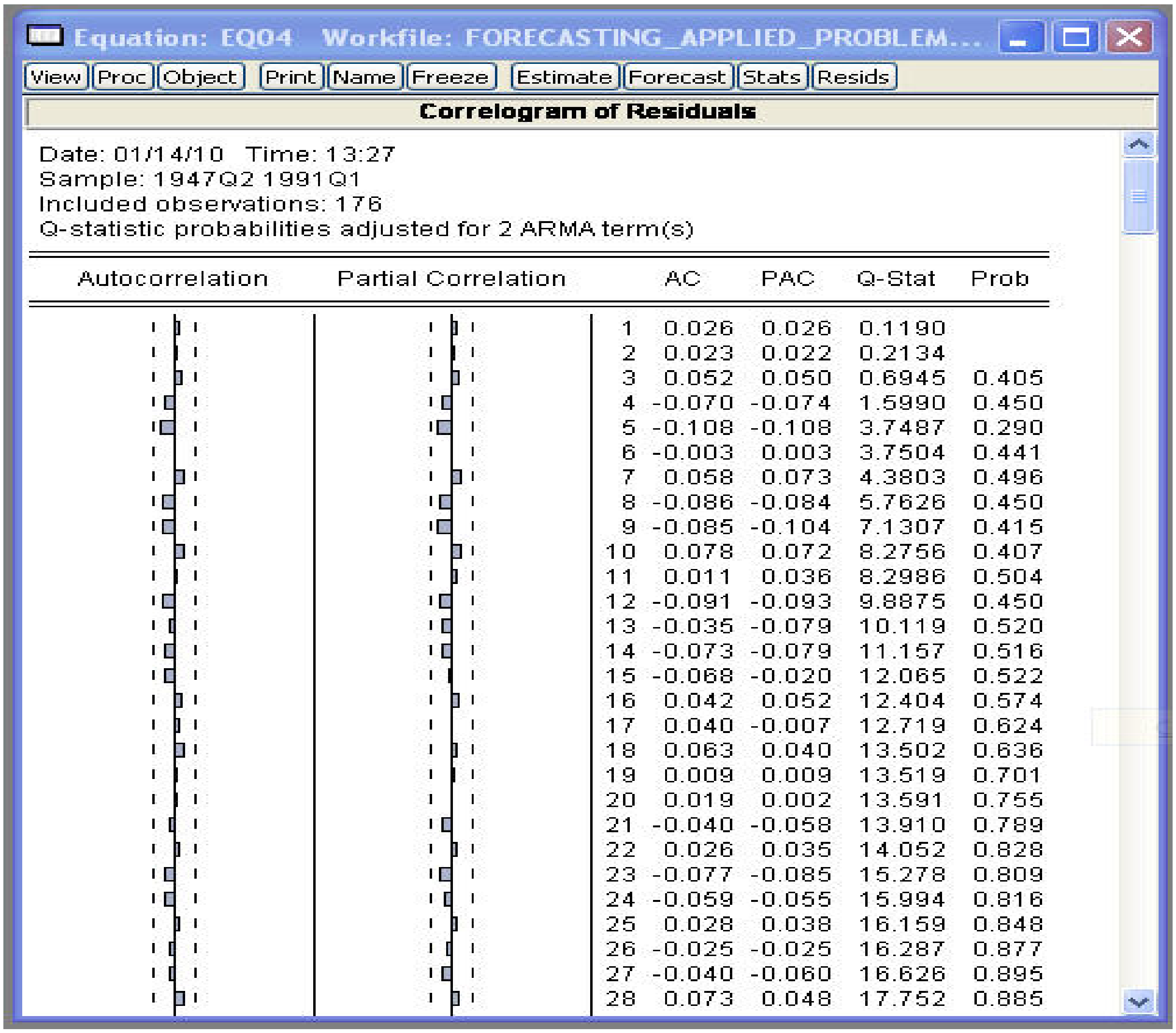


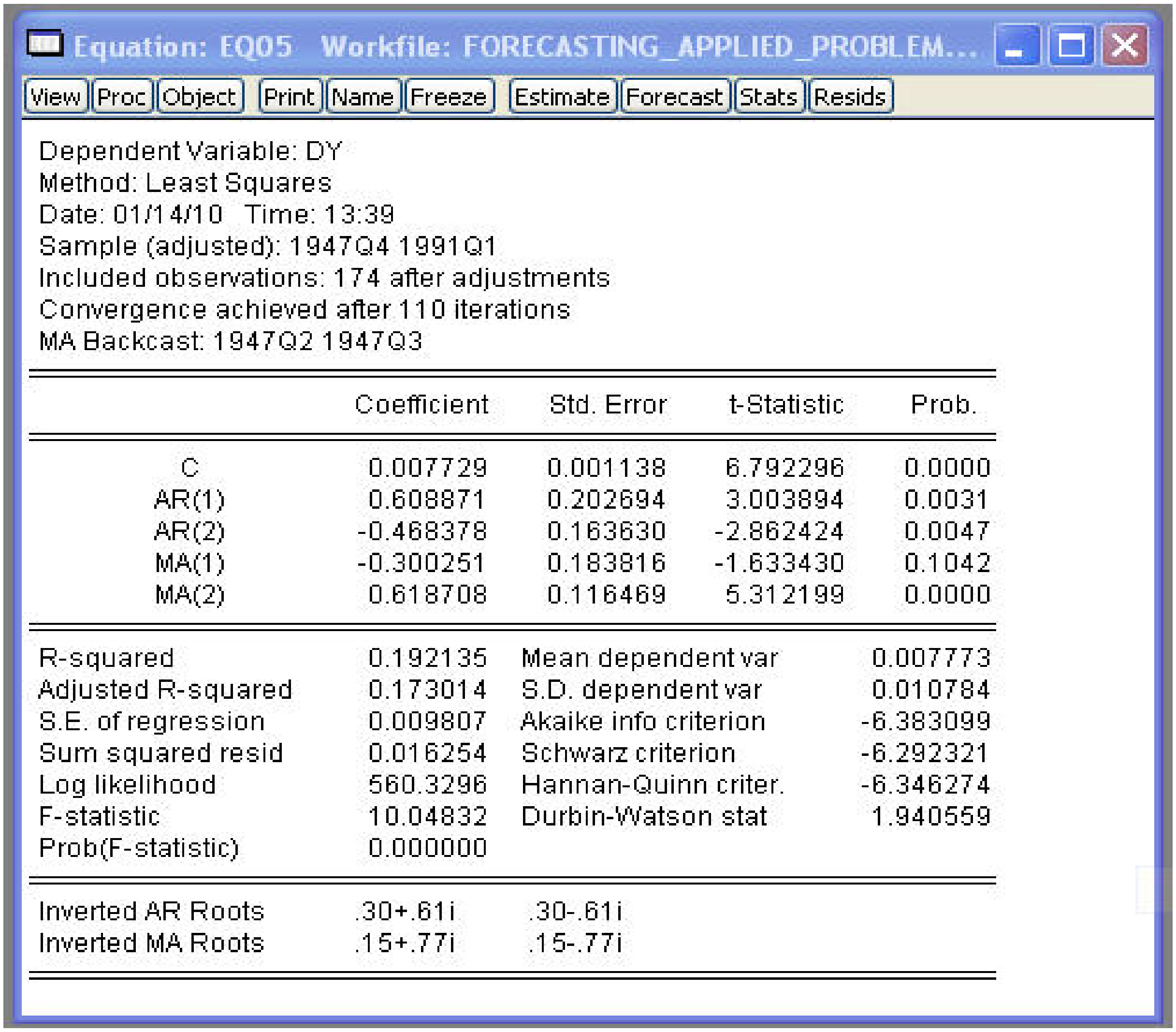
Figure 18: The MA(2) model ACF/data SACF and model PACF/data SPACF. The model-implied ACF and PACF do not successfully capture the dependence in the data.



Date: 01/14/10 Time: 13:27
Sample: 1947Q2 1991Q1
Included observations: 176
Q-statistic probabilities adjusted for 2 ARMA term(s)

Lag	AC	PAC	Q-Stat	Prob
1	0.026	0.026	0.1190	
2	0.023	0.022	0.2134	
3	0.052	0.050	0.6945	0.405
4	-0.070	-0.074	1.5990	0.450
5	-0.108	-0.108	3.7487	0.290
6	-0.003	0.003	3.7504	0.441
7	0.058	0.073	4.3803	0.496
8	-0.086	-0.084	5.7626	0.450
9	-0.085	-0.104	7.1307	0.415
10	0.078	0.072	8.2756	0.407
11	0.011	0.036	8.2986	0.504
12	-0.091	-0.093	9.8875	0.450
13	-0.035	-0.079	10.119	0.520
14	-0.073	-0.079	11.157	0.516
15	-0.068	-0.020	12.065	0.522
16	0.042	0.052	12.404	0.574
17	0.040	-0.007	12.719	0.624
18	0.063	0.040	13.502	0.636
19	0.009	0.009	13.519	0.701
20	0.019	0.002	13.591	0.755
21	-0.040	-0.058	13.910	0.789
22	0.026	0.035	14.052	0.828
23	-0.077	-0.085	15.278	0.809
24	-0.059	-0.055	15.994	0.816
25	0.028	0.038	16.159	0.848
26	-0.025	-0.025	16.287	0.877
27	-0.040	-0.060	16.626	0.895
28	0.073	0.048	17.752	0.885

Figure 19: ACF and PACF and Q-statistics for estimated residuals from the MA(2) model fitted to Δy_t, indicating that the residuals are approximately white noise. However, the poor fit of the oscillatory decay in the ACF/PACF again suggests that a better model can be found.



Equation: EQ05 Workfile: FORECASTING_APPLIED_PROBLEM...

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: DY
Method: Least Squares
Date: 01/14/10 Time: 13:39
Sample (adjusted): 1947Q4 1991Q1
Included observations: 174 after adjustments
Convergence achieved after 110 iterations
MA Backcast: 1947Q2 1947Q3

	Coefficient	Std. Error	t-Statistic	Prob.
C	0.007729	0.001138	6.792296	0.0000
AR(1)	0.608871	0.202694	3.003894	0.0031
AR(2)	-0.468378	0.163630	-2.862424	0.0047
MA(1)	-0.300251	0.183816	-1.633430	0.1042
MA(2)	0.618708	0.116469	5.312199	0.0000

R-squared	0.192135	Mean dependent var	0.007773
Adjusted R-squared	0.173014	S.D. dependent var	0.010784
S.E. of regression	0.009807	Akaike info criterion	-6.383099
Sum squared resid	0.016254	Schwarz criterion	-6.292321
Log likelihood	560.3296	Hannan-Quinn criter.	-6.346274
F-statistic	10.04832	Durbin-Watson stat	1.940559
Prob(F-statistic)	0.000000		

Inverted AR Roots	.30+.61i	.30-.61i
Inverted MA Roots	.15+.77i	.15-.77i

Figure 20: ARMA(2,2) regression $\Delta y_t = \alpha_0 + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \beta_2 \varepsilon_{t-2}$, with $R^2 \approx 0.19$, generally significant coefficients, 'good' Durbin-Watson statistic, and stationary characteristics.

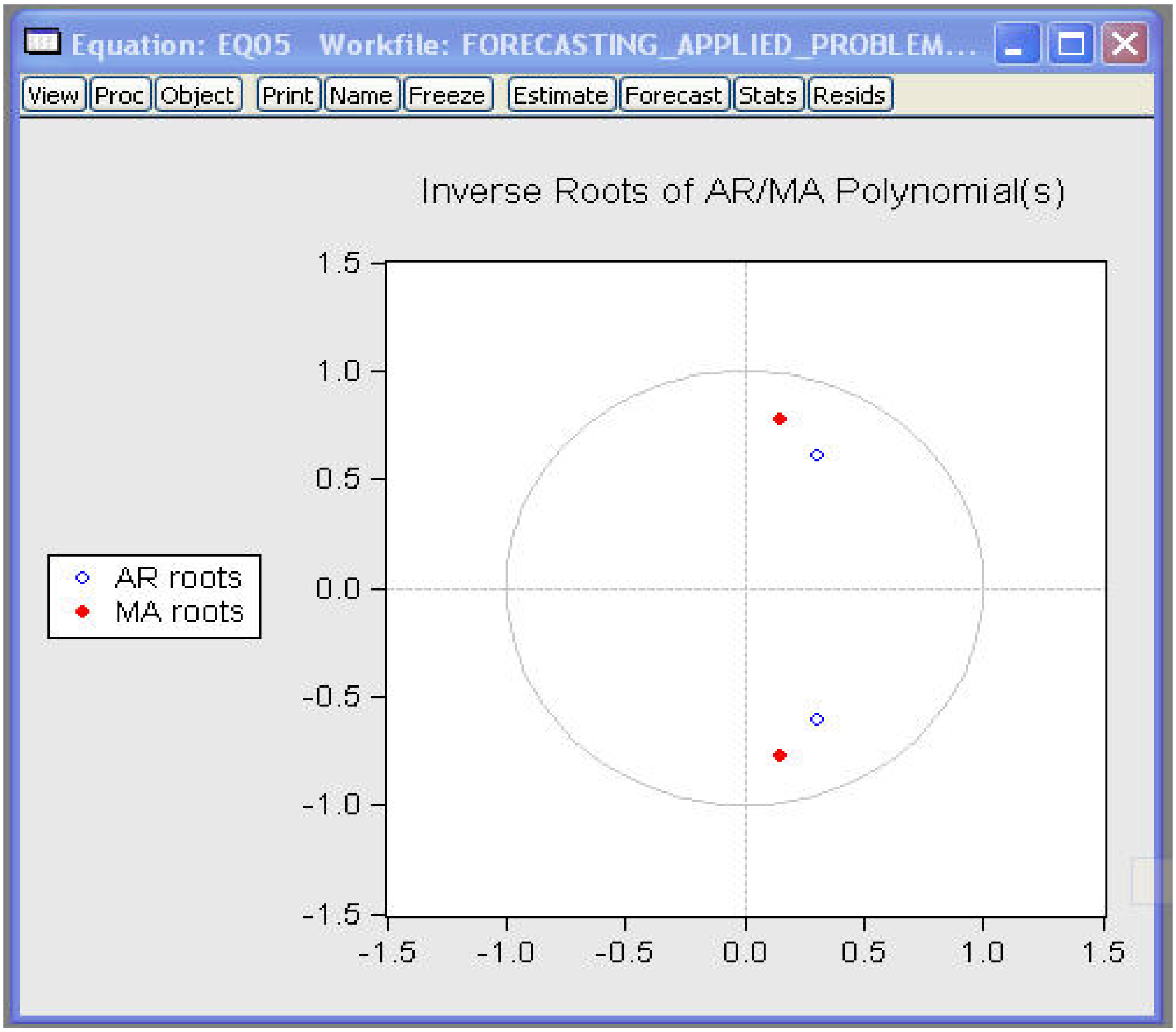


Figure 21: Roots of ARMA(2,2) inverse polynomials, indicating that the model is stationary and invertible.

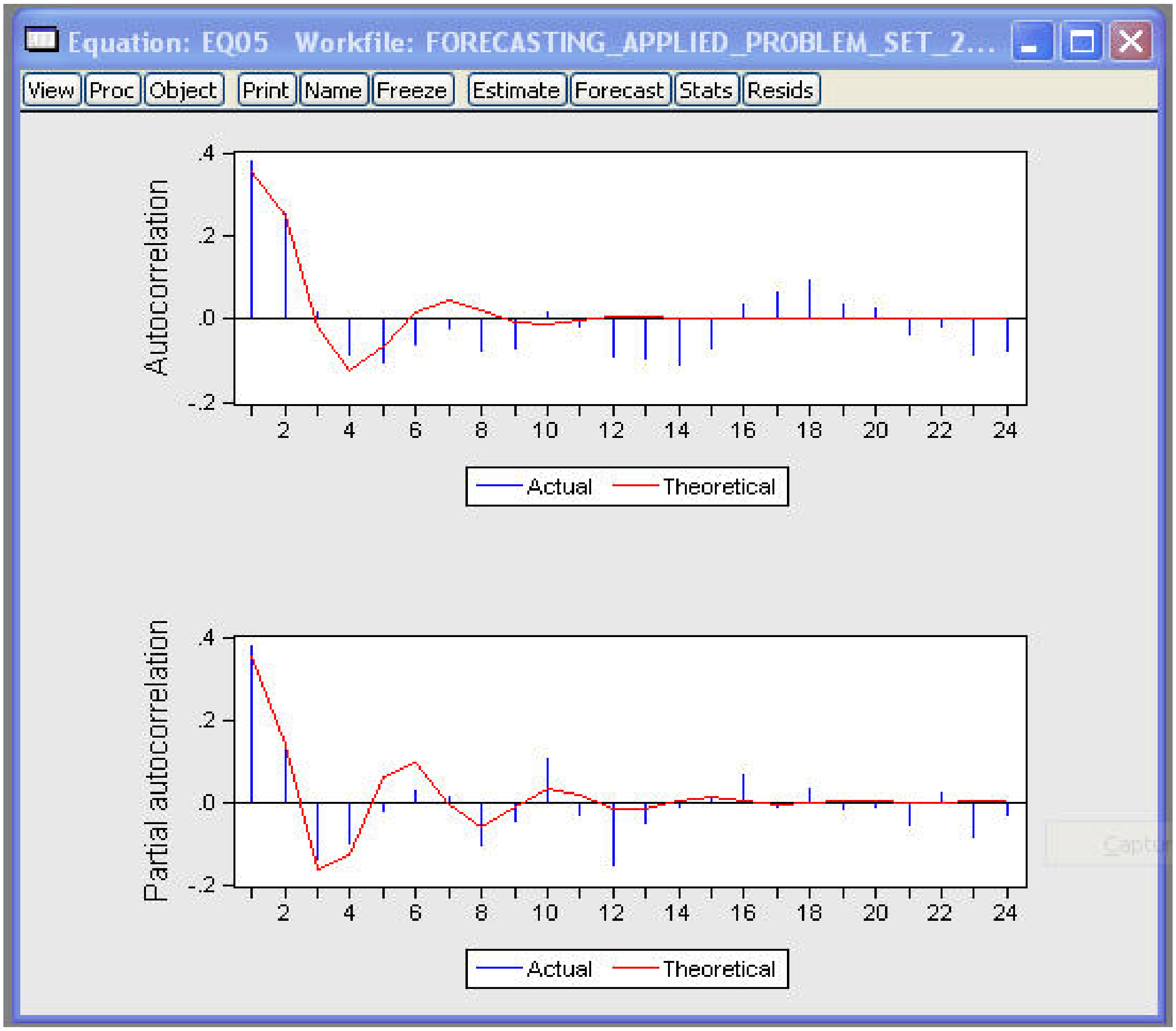
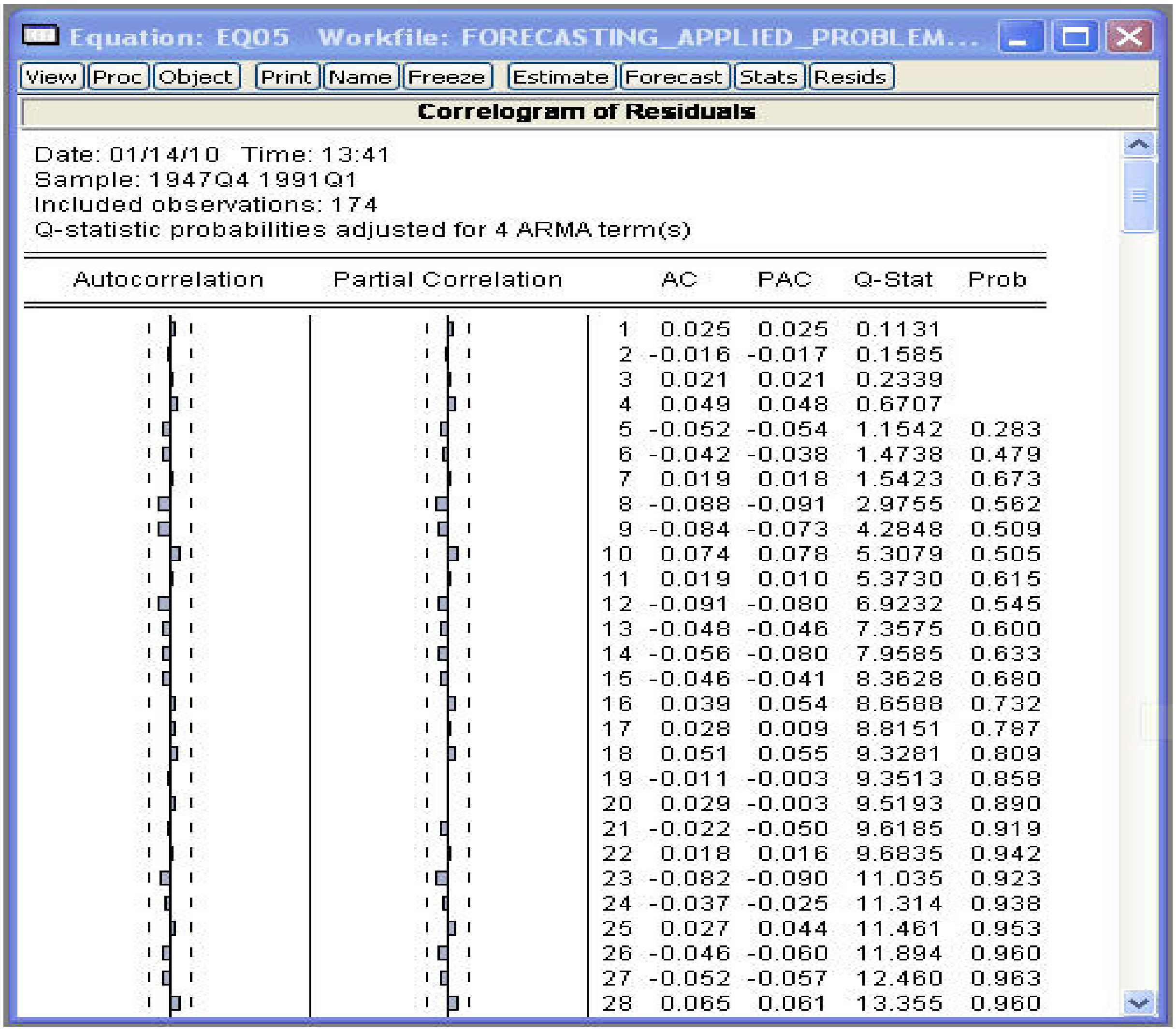


Figure 22: The ARMA(2,2) model ACF/data SACF and model PACF/data SPACF. The model-implied ACF and PACF capture the dependence in the data reasonably well (and this is *as* important as the model minimizing SIC across p, q).



Equation: EQ05 Workfile: FORECASTING_APPLIED_PROBLEM...

View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Correlogram of Residuals

Date: 01/14/10 Time: 13:41
Sample: 1947Q4 1991Q1
Included observations: 174
Q-statistic probabilities adjusted for 4 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.025	0.025	0.1131	
		2	-0.016	-0.017	0.1585	
		3	0.021	0.021	0.2339	
		4	0.049	0.048	0.6707	
		5	-0.052	-0.054	1.1542	0.283
		6	-0.042	-0.038	1.4738	0.479
		7	0.019	0.018	1.5423	0.673
		8	-0.088	-0.091	2.9755	0.562
		9	-0.084	-0.073	4.2848	0.509
		10	0.074	0.078	5.3079	0.505
		11	0.019	0.010	5.3730	0.615
		12	-0.091	-0.080	6.9232	0.545
		13	-0.048	-0.046	7.3575	0.600
		14	-0.056	-0.080	7.9585	0.633
		15	-0.046	-0.041	8.3628	0.680
		16	0.039	0.054	8.6588	0.732
		17	0.028	0.009	8.8151	0.787
		18	0.051	0.055	9.3281	0.809
		19	-0.011	-0.003	9.3513	0.858
		20	0.029	-0.003	9.5193	0.890
		21	-0.022	-0.050	9.6185	0.919
		22	0.018	0.016	9.6835	0.942
		23	-0.082	-0.090	11.035	0.923
		24	-0.037	-0.025	11.314	0.938
		25	0.027	0.044	11.461	0.953
		26	-0.046	-0.060	11.894	0.960
		27	-0.052	-0.057	12.460	0.963
		28	0.065	0.061	13.355	0.960

Figure 23: ACF and PACF and Q-statistics for estimated residuals from the ARMA(2,2) model fitted to Δy_t, indicating that the residuals are approximately white noise. This model could be used for simple short-term forecasting.

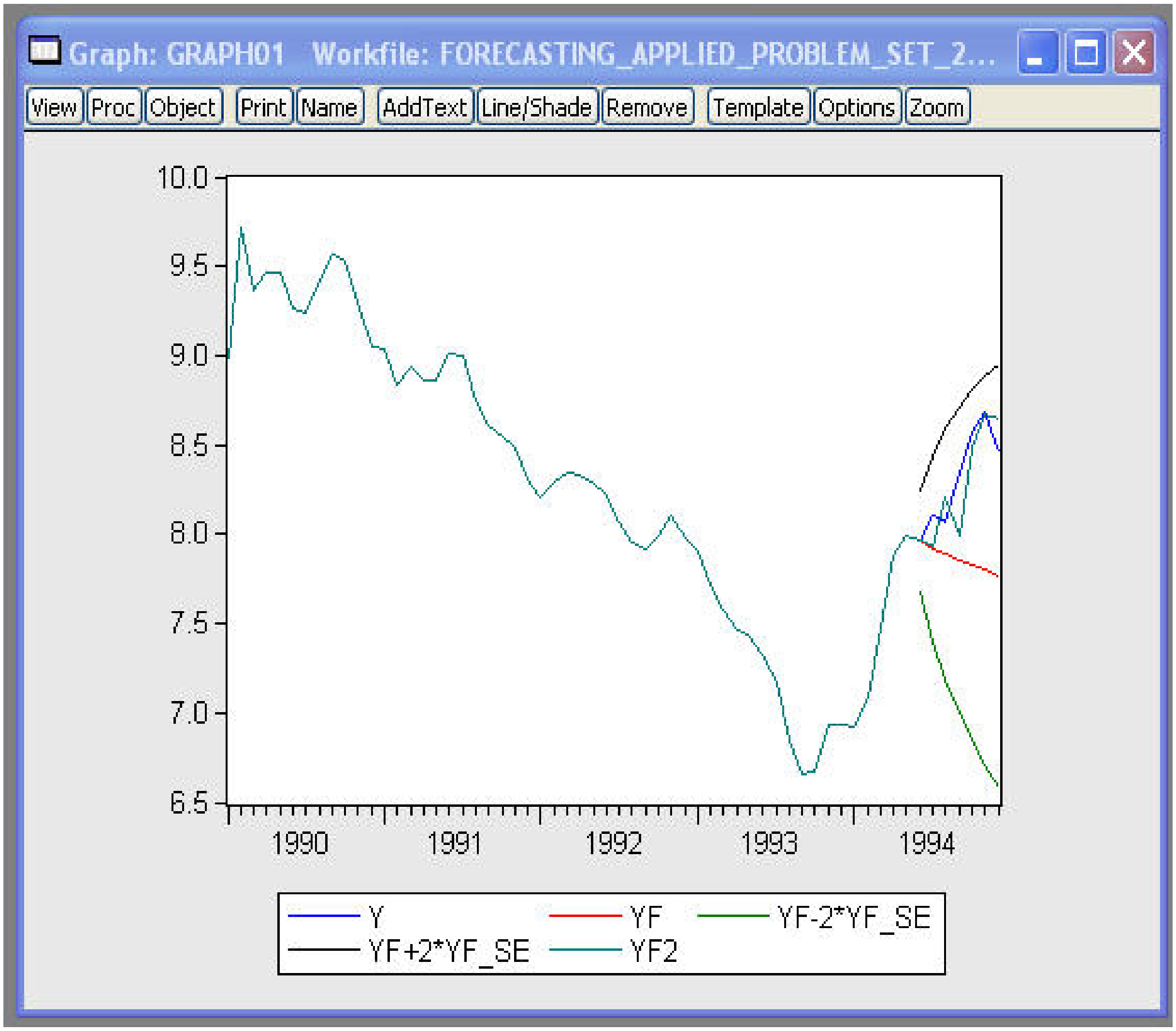


Figure 24: ARIMA(1,1,2) short-term forecasts (dynamic forecast and error bands, and static forecast) for y_t (bond yields).

Areas Under the Normal Curve

Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p
0.00	0.5000	0.5000	0.40	0.6554	0.3446	0.80	0.7881	0.2119	1.20	0.8849	0.1151	1.60	0.9452	0.0548
0.01	0.5040	0.4960	0.41	0.6591	0.3409	0.81	0.7910	0.2090	1.21	0.8869	0.1131	1.61	0.9463	0.0537
0.02	0.5080	0.4920	0.42	0.6628	0.3372	0.82	0.7939	0.2061	1.22	0.8888	0.1112	1.62	0.9474	0.0526
0.03	0.5120	0.4880	0.43	0.6664	0.3336	0.83	0.7967	0.2033	1.23	0.8907	0.1093	1.63	0.9484	0.0516
0.04	0.5160	0.4840	0.44	0.6700	0.3300	0.84	0.7995	0.2005	1.24	0.8925	0.1075	1.64	0.9495	0.0505
0.05	0.5199	0.4801	0.45	0.6736	0.3264	0.85	0.8023	0.1977	1.25	0.8944	0.1056	**1.65**	**0.9505**	**0.0495**
0.06	0.5239	0.4761	0.46	0.6772	0.3228	0.86	0.8051	0.1949	1.26	0.8962	0.1038	1.66	0.9515	0.0485
0.07	0.5279	0.4721	0.47	0.6808	0.3192	0.87	0.8078	0.1922	1.27	0.8980	0.1020	1.67	0.9525	0.0475
0.08	0.5319	0.4681	0.48	0.6844	0.3156	0.88	0.8106	0.1894	1.28	0.8997	0.1003	1.68	0.9535	0.0465
0.09	0.5359	0.4641	0.49	0.6879	0.3121	0.89	0.8133	0.1867	1.29	0.9015	0.0985	1.69	0.9545	0.0455
0.10	0.5398	0.4602	0.50	0.6915	0.3085	0.90	0.8159	0.1841	1.30	0.9032	0.0968	1.70	0.9554	0.0446
0.11	0.5438	0.4562	0.51	0.6950	0.3050	0.91	0.8186	0.1814	1.31	0.9049	0.0951	1.71	0.9564	0.0436
0.12	0.5478	0.4522	0.52	0.6985	0.3015	0.92	0.8212	0.1788	1.32	0.9066	0.0934	1.72	0.9573	0.0427
0.13	0.5517	0.4483	0.53	0.7019	0.2981	0.93	0.8238	0.1762	1.33	0.9082	0.0918	1.73	0.9582	0.0418
0.14	0.5557	0.4443	0.54	0.7054	0.2946	0.94	0.8264	0.1736	1.34	0.9099	0.0901	1.74	0.9591	0.0409
0.15	0.5596	0.4404	0.55	0.7088	0.2912	0.95	0.8289	0.1711	1.35	0.9115	0.0885	1.75	0.9599	0.0401
0.16	0.5636	0.4364	0.56	0.7123	0.2877	0.96	0.8315	0.1685	1.36	0.9131	0.0869	1.76	0.9608	0.0392
0.17	0.5675	0.4325	0.57	0.7157	0.2843	0.97	0.8340	0.1660	1.37	0.9147	0.0853	1.77	0.9616	0.0384
0.18	0.5714	0.4286	0.58	0.7190	0.2810	0.98	0.8365	0.1635	1.38	0.9162	0.0838	1.78	0.9625	0.0375
0.19	0.5753	0.4247	0.59	0.7224	0.2776	0.99	0.8389	0.1611	1.39	0.9177	0.0823	1.79	0.9633	0.0367
0.20	0.5793	0.4207	0.60	0.7257	0.2743	1.00	0.8413	0.1587	1.40	0.9192	0.0808	1.80	0.9641	0.0359
0.21	0.5832	0.4168	0.61	0.7291	0.2709	1.01	0.8438	0.1562	1.41	0.9207	0.0793	1.81	0.9649	0.0351
0.22	0.5871	0.4129	0.62	0.7324	0.2676	1.02	0.8461	0.1539	1.42	0.9222	0.0778	1.82	0.9656	0.0344
0.23	0.5910	0.4090	0.63	0.7357	0.2643	1.03	0.8485	0.1515	1.43	0.9236	0.0764	1.83	0.9664	0.0336
0.24	0.5948	0.4052	0.64	0.7389	0.2611	1.04	0.8508	0.1492	1.44	0.9251	0.0749	1.84	0.9671	0.0329
0.25	0.5987	0.4013	0.65	0.7422	0.2578	1.05	0.8531	0.1469	1.45	0.9265	0.0735	1.85	0.9678	0.0322
0.26	0.6026	0.3974	0.66	0.7454	0.2546	1.06	0.8554	0.1446	1.46	0.9279	0.0721	1.86	0.9686	0.0314
0.27	0.6064	0.3936	0.67	0.7486	0.2514	1.07	0.8577	0.1423	1.47	0.9292	0.0708	1.87	0.9693	0.0307
0.28	0.6103	0.3897	0.68	0.7517	0.2483	1.08	0.8599	0.1401	1.48	0.9306	0.0694	1.88	0.9699	0.0301
0.29	0.6141	0.3859	0.69	0.7549	0.2451	1.09	0.8621	0.1379	1.49	0.9319	0.0681	1.89	0.9706	0.0294
0.30	0.6179	0.3821	0.70	0.7580	0.2420	1.10	0.8643	0.1357	1.50	0.9332	0.0668	1.90	0.9713	0.0287
0.31	0.6217	0.3783	0.71	0.7611	0.2389	1.11	0.8665	0.1335	1.51	0.9345	0.0655	1.91	0.9719	0.0281
0.32	0.6255	0.3745	0.72	0.7642	0.2358	1.12	0.8686	0.1314	1.52	0.9357	0.0643	1.92	0.9726	0.0274
0.33	0.6293	0.3707	0.73	0.7673	0.2327	1.13	0.8708	0.1292	1.53	0.9370	0.0630	1.93	0.9732	0.0268
0.34	0.6331	0.3669	0.74	0.7704	0.2296	1.14	0.8729	0.1271	1.54	0.9382	0.0618	1.94	0.9738	0.0262
0.35	0.6368	0.3632	0.75	0.7734	0.2266	1.15	0.8749	0.1251	1.55	0.9394	0.0606	1.95	0.9744	0.0256
0.36	0.6406	0.3594	0.76	0.7764	0.2236	1.16	0.8770	0.1230	1.56	0.9406	0.0594	**1.96**	**0.9750**	**0.0250**
0.37	0.6443	0.3557	0.77	0.7794	0.2206	1.17	0.8790	0.1210	1.57	0.9418	0.0582	1.97	0.9756	0.0244
0.38	0.6480	0.3520	0.78	0.7823	0.2177	1.18	0.8810	0.1190	1.58	0.9429	0.0571	1.98	0.9761	0.0239
0.39	0.6517	0.3483	0.79	0.7852	0.2148	1.19	0.8830	0.1170	1.59	0.9441	0.0559	1.99	0.9767	0.0233

Figure 25: Statistical table for N(0, 1).

Critical Values of the t Distribution

df	2-tailed testing			1-tailed testing		
	• •			• •		
	0.1	0.05	0.01	0.1	0.05	0.01
5	2.015	2.571	4.032	1.476	2.015	3.365
6	1.943	2.447	3.707	1.440	1.943	3.143
7	1.895	2.365	3.499	1.415	1.895	2.998
8	1.860	2.306	3.355	1.397	1.860	2.896
9	1.833	2.262	3.250	1.383	1.833	2.821
10	1.812	2.228	3.169	1.372	1.812	2.764
11	1.796	2.201	3.106	1.363	1.796	2.718
12	1.782	2.179	3.055	1.356	1.782	2.681
13	1.771	2.160	3.012	1.350	1.771	2.650
14	1.761	2.145	2.977	1.345	1.761	2.624
15	1.753	2.131	2.947	1.341	1.753	2.602
16	1.746	2.120	2.921	1.337	1.746	2.583
17	1.740	2.110	2.898	1.333	1.740	2.567
18	1.734	2.101	2.878	1.330	1.734	2.552
19	1.729	2.093	2.861	1.328	1.729	2.539
20	1.725	2.086	2.845	1.325	1.725	2.528
21	1.721	2.080	2.831	1.323	1.721	2.518
22	1.717	2.074	2.819	1.321	1.717	2.508
23	1.714	2.069	2.807	1.319	1.714	2.500
24	1.711	2.064	2.797	1.318	1.711	2.492
25	1.708	2.060	2.787	1.316	1.708	2.485
26	1.706	2.056	2.779	1.315	1.706	2.479
27	1.703	2.052	2.771	1.314	1.703	2.473
28	1.701	2.048	2.763	1.313	1.701	2.467
29	1.699	2.045	2.756	1.311	1.699	2.462
30	1.697	2.042	2.750	1.310	1.697	2.457
40	1.684	2.021	2.704	1.303	1.684	2.423
50	1.676	2.009	2.678	1.299	1.676	2.403
60	1.671	2.000	2.660	1.296	1.671	2.390
80	1.664	1.990	2.639	1.292	1.664	2.374
100	1.660	1.984	2.626	1.290	1.660	2.364
120	1.658	1.980	2.617	1.289	1.658	2.358
• •	1.645	1.960	2.576	1.282	1.645	2.327

Figure 26: Statistical table for Student's $t(r)$.

Critical Values of the <u>F</u> Distribution
(α = .05)

df	df between										
within	1	2	3	4	5	6	7	8	12	24	∞
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.68	4.53	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.79	2.61	2.41
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.69	2.51	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.20	2.01	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.18	1.98	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.16	1.96	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.12	1.91	1.66
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	1.92	1.70	1.39
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	1.88	1.65	1.33
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.85	1.63	1.28
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.83	1.61	1.26
∞	3.84	3.00	2.61	2.37	2.22	2.10	2.01	1.94	1.75	1.52	1.00

Figure 27: Statistical table for $F(m, n)$ at the 5% level.

Critical Values of the F Distribution
($\alpha = .01$)

df	df between										
within	1	2	3	4	5	6	7	8	12	24	∞
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	9.89	9.47	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.71	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	3.96	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	3.80	3.43	3.01
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.46	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.17	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	2.90	2.52	2.07
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	2.87	2.49	2.04
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.66	2.29	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.50	2.12	1.60
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.42	2.03	1.50
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.37	1.98	1.43
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.34	1.95	1.38
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.19	1.79	1.00

Figure 28: Statistical table for $F(m, n)$ at the 1% level.

Critical Values of the χ^2 Distribution

df	Area in the Upper Tail					
	0.99	0.95	0.9	0.1	**0.05**	**0.01**
1	0.000	0.004	0.016	2.706	3.841	6.635
2	0.020	0.103	0.211	4.605	5.991	9.210
3	0.115	0.352	0.584	6.251	7.815	11.345
4	0.297	0.711	1.064	7.779	9.488	13.277
5	0.554	1.145	1.610	9.236	11.070	15.086
6	0.872	1.635	2.204	10.645	12.592	16.812
7	1.239	2.167	2.833	12.017	14.067	18.475
8	1.646	2.733	3.490	13.362	15.507	20.090
9	2.088	3.325	4.168	14.684	16.919	21.666
10	2.558	3.940	4.865	15.987	18.307	23.209
11	3.053	4.575	5.578	17.275	19.675	24.725
12	3.571	5.226	6.304	18.549	21.026	26.217
13	4.107	5.892	7.042	19.812	22.362	27.688
14	4.660	6.571	7.790	21.064	23.685	29.141
15	5.229	7.261	8.547	22.307	24.996	30.578
16	5.812	7.962	9.312	23.542	26.296	32.000
17	6.408	8.672	10.085	24.769	27.587	33.409
18	7.015	9.390	10.865	25.989	28.869	34.805
19	7.633	10.117	11.651	27.204	30.144	36.191
20	8.260	10.851	12.443	28.412	31.410	37.566
21	8.897	11.591	13.240	29.615	32.671	38.932
22	9.542	12.338	14.041	30.813	33.924	40.289
23	10.196	13.091	14.848	32.007	35.172	41.638
24	10.856	13.848	15.659	33.196	36.415	42.980
25	11.524	14.611	16.473	34.382	37.652	44.314

Figure 29: Statistical table for $\chi^2(q)$.