Discrete choice modelling

- This problem set deals with maximum-likelihood estimation and asymptotic inference for binary choice models applied to cross-sectional data.
- We use the grade data from Spector and Mazzeo (1980, Journal of Economic Education), that is available as `grades.txt` on the website.
- There are $n = 32$ observations (obs) on individuals' grade point average (gpa), score on a pretest that indicates initial knowledge of the subject matter (tuce), an indicator of exposure to a new teaching method (psi), and an indicator of whether examination grades improved (1) or worsened (0) (grade).
- An important question from a policy-making standpoint is: **do grades improve** (grade $= 1$) **after exposure to the new teaching method** (psi $= 1$)**?**
- Refer to figures 1–33, and attempt the following:
  1. Import the data, and perform a careful preliminary analysis of the variables. Briefly examine scatter plots, and simple descriptive statistics, to gain intuition about the behaviour of the data. What features of interest do you see?

     What is the range of the grade point average and the pretest score? What percentage of the individuals had an improved grade? What percentage of the individuals had *not* been exposed to the new teaching method before the exam?

2. Estimate a probit model (probability of grade improvement as a function of explanatory variables: constant, grade point average, score on pretest, and exposure to new teaching method), and label the equation "probit_eqn":

$$\text{Prob(grade} = 1) = \Phi(\beta_0 + \beta_1 \text{ gpa} + \beta_2 \text{ tuce} + \beta_3 \text{ psi}).$$

What are the *signs* of the estimated coefficients? Let y refer to the dependent variable throughout, and x to the vector of explanatory variables (constant, gpa, tuce, and psi). The marginal effects are:

$$\frac{\partial \text{Prob}(y = 1)}{\partial x} = \frac{\partial \text{F}(x'\beta)}{\partial x} = f(x'\beta)\ \beta,$$

where for the probit model, $\text{F}(\cdot) = \Phi(\cdot)$, the normal N(0,1) cumulative distribution function, and $f(\cdot) = \phi(\cdot)$ is the density function. For obs = 1 and obs = 10, compute the marginal effects manually.

What do you notice about $\partial \widehat{\text{Prob}}(y = 1)/\partial x$ (compare the results for obs = 1 and obs = 10 to answer this question)? [This illustrates one of the difficulties of interpreting marginal effects with binary choice models.]

Compute the sample means of gpa, tuce and psi. Then, manually calculate $\partial \widehat{\text{Prob}}(y = 1)/\partial x$ given that $x = \bar{x}$ (the sample mean), where the sample mean of the constant is simply 1. Interpret your result.

3. Estimate a probit model of grade on constant, gpa, tuce and psi, using the Huber-White robust asymptotic covariance matrix $\widehat{\text{AVar}}(\widehat{\beta}) = \widehat{H}^{-1}\widehat{B}\widehat{H}^{-1} =$

$$\left[\frac{\partial^2 \ln L}{\partial\beta\partial\beta'}\bigg|\beta = \widehat{\beta}\right]^{-1} \left[\sum_{i=1}^{n}\left(\frac{\phi(x_i'\beta)(y_i - \Phi(x_i'\beta))}{\Phi(x_i'\beta)(1-\Phi(x_i'\beta))}\right)^2 x_i x_i'\bigg|\beta = \widehat{\beta}\right] \left[\frac{\partial^2 \ln L}{\partial\beta\partial\beta'}\bigg|\beta = \widehat{\beta}\right]^{-1},$$

and name this equation "probit_eqn_robust". What are the numerical values of the estimated standard errors $\widehat{\text{se}}(\widehat{\beta}) := (\widehat{\text{AVar}}(\widehat{\beta}))^{1/2}$?

4. The asymptotic covariance matrix of the estimated *probabilities* $\widehat{\text{Prob}}(y = 1) = F(x'\widehat{\beta})$ are given by the "delta method" as (with an abuse of notation):

$$\widehat{\text{AVar}}(F(x'\widehat{\beta})) = \left(\frac{\partial F(x'\widehat{\beta})}{\partial\widehat{\beta}'}\right)\widehat{\text{AVar}}(\widehat{\beta})\left(\frac{\partial F(x'\widehat{\beta})}{\partial\widehat{\beta}'}\right)',$$

where

$$\frac{\partial F(x'\widehat{\beta})}{\partial\widehat{\beta}} = \left(\frac{\partial F(x'\widehat{\beta})}{\partial(x'\widehat{\beta})}\right)\left(\frac{\partial(x'\widehat{\beta})}{\partial\widehat{\beta}}\right) = f(x'\widehat{\beta})\ x,$$

so that

$$\widehat{\text{AVar}}(F(x'\widehat{\beta})) = (f(x'\widehat{\beta}))^2 x'\widehat{\text{AVar}}(\widehat{\beta})x,$$

which depends upon $f(\cdot)$ and x.

Set $x = \bar{x}$ (the sample mean of the explanatory variables). Then, using the *matrix* $\widehat{\text{AVar}}(\widehat{\beta})$ from part 3 (see figures 7 and 8)), with $f(\cdot)$ and $F(\cdot)$ the density and distribution functions of the normal N(0,1), compute the *scalar* $\widehat{\text{AVar}}(F(\bar{x}'\widehat{\beta}))$ [Hint: this is equal to 0.0791365 times $(f(\bar{x}'\widehat{\beta}))^2$].

5. Using the robust standard errors from $\widehat{\text{AVar}}(\widehat{\beta})$, test the null hypothesis H_0 : $\beta_2 = 0$ (the coefficient on tuce), against the two-sided alternative, at the 95% level of significance. Use both a t test (compute this yourself, and justify your choice of critical value), and a Wald test (compute this using the software).

6. Plot the estimated residuals and the fitted probabilities. Does this give you any intuition regarding the quality of the probit model fit to the data? How would you test whether the probability $\text{Prob}(y = 1)$ is equal to a constant, against a one-sided alternative? (You do not need to perform the test).

7. Compute the likelihood ratio statistic:

$$\text{LR} = -2(\ln \widehat{L}_0 - \ln \widehat{L}) \sim \chi^2(q),$$

where $\ln \widehat{L}$ is the maximized log-likelihood, and $\ln \widehat{L}_0$ is the maximized log-likelihood under the restriction that all coefficients except the constant are zero, and q is the number of restrictions imposed under the null that the restricted model is true. Test the null hypothesis that the restricted model is true, against the alternative that it is not, at the 95% level of significance.

8. Show that:

$$\ln \widehat{L}_0 = n[\bar{y} \ln \bar{y} + (1 - \bar{y}) \ln(1 - \bar{y})],$$

where $\bar{y}$ is the proportion of all observations that have dependent variable equal to 1 (we saw in part 1 that $\bar{y} \approx 0.343750$, i.e., the proportion of students with a grade improvement (grade = 1)).

Compute $\ln \widehat{L}_0$, and compare the result with that in figure 10.

9. McFadden's pseudo-R^2 is defined as:

$$R^2_M = 1 - \frac{\ln \widehat{L}}{\ln \widehat{L}_0}.$$

Compute this, and compare with the result in figure 10. Why is $R^2_M < 1$?

10. Estimate a logit model (probability of grade improvement as a function of explanatory variables: constant, grade point average, score on pretest, and exposure to new teaching method), and label the equation "logit_eqn":

$$\text{Prob}(\text{grade} = 1) = \Lambda(\beta_0 + \beta_1 \text{ gpa} + \beta_2 \text{ tuce} + \beta_3 \text{ psi}).$$

What are the *signs* of the estimated coefficients? Note that:

$$\frac{\partial \text{Prob}(y = 1)}{\partial x} = \frac{\partial \text{F}(x'\beta)}{\partial x} = f(x'\beta)\ \beta,$$

where for the logit model, $\text{F}(x'\beta) = \Lambda(x'\beta) = (e^{x'\beta})/(1 + e^{x'\beta})$, the standard logistic cumulative distribution function. We have seen that:

$$\frac{\partial \widehat{\text{Prob}}(y = 1)}{\partial x} = \Lambda(x'\widehat{\beta})[1 - \Lambda(x'\widehat{\beta})]\widehat{\beta}.$$

Calculate the marginal effects, given that $x = \bar{x}$ (the sample mean), where we note that the sample mean of the constant is simply 1.

11. For the logit model, the probability $p := \text{Prob}(y = 1)$ is defined as:

$$p := \frac{e^{x'\beta}}{1 + e^{x'\beta}}.$$

Show that:

$$\frac{p}{1-p} = e^{x'\beta}.$$

This quantity is known as the *odds-ratio*, and can be useful when interpreting coefficients: it measures the probability of $y = 1$ relative to the probability of $y = 0$. For instance, if $p/(1-p) = 2$, then the 'odds' of (chance of) obtaining $y = 1$ are twice those of $y = 0$.

For the logit model in figure 11, evaluate the estimated odds ratio $\widehat{p}/(1-\widehat{p})$ at $x = \bar{x}$ (the sample mean).

12. Plot the estimated residuals and the fitted probabilities. Does this give you any intuition regarding the quality of the logit model fit to the data? Consider figure 12 (compare the fitted and actual probabilities, in particular) with your answer to part 11 in mind, and explain your findings.

13. Another interpretation for the logit model slope coefficients is that, if the jth regressor increases by one unit, then $e^{x'\beta}$ increases to $e^{x'\beta}e^{\beta_j}$, and so $p/(1-p) = e^{x'\beta}$ has increased by a factor e^{β_j}. For the model in figure 11, interpret the slope coefficient $\widehat{\beta}_1 \approx 2.83$ in this way. [The resultant relative chance of obtaining $y = 1$ may seem huge.] Check the range of actual values of gpa, and comment.

    The original odds-ratio for the logit model was $\widehat{p}/(1-\widehat{p}) \approx 0.338368$. Use your answer to the last question (on $e^{\widehat{\beta}_1}$) to compute the new odds-ratio. What does this imply about the probability of grade improvement ($= p$) after a unit increase in gpa? [Hint: show that p is now 0.84, i.e., a unit increase in gpa (which is considerable), holding everything else constant, more than doubles the probability that $y = 1$.]

14. Since the marginal effects will depend upon the explanatory variables in general, refer to figures 13–33, to plot a probability response curve $\widehat{\text{Prob}}(y = 1)$ against gpa, for psi = 0 and psi = 1, for both the probit model (figure 4) and the logit model (figure 11). That is, what is the probability of a grade increase given participation (or not) in the new teaching method, conditional on gpa (roughly, "intelligence"), *holding all other variables at their sample means*? [Hint: What is the effect of psi = 1 on the probabilities? What is the marginal effect of psi? What do you notice about the probability of grade increase after exposure to the new method, conditional on gpa score?]

15. Check the rule-of-thumb (which holds approximately):

$$\widehat{\beta}_{\texttt{LOGIT}} \approx \left(\frac{\pi}{\sqrt{3}}\right) \widehat{\beta}_{\texttt{PROBIT}}$$

using the estimated coefficients from figures 4 and 11.

16. Manually construct a 2×2 table of percentage "hits and misses", as a summary of the predictive ability of "probit_eqn". Use the decision rule: if $\widehat{\Phi} > 0.5$, $\widehat{\text{grade}} = 1$; and if $\widehat{\Phi} \leq 0.5$, $\widehat{\text{grade}} = 0$. What do you notice?

OBS	GPA	TUCE	PSI	GRADE
1	2.66	20	0	0
2	2.89	22	0	0
3	3.28	24	0	0
4	2.92	12	0	0
5	4.00	21	0	1
6	2.86	17	0	0
7	2.76	17	0	0
8	2.87	21	0	0
9	3.03	25	0	0
10	3.92	29	0	1
11	2.63	20	0	0
12	3.32	23	0	0
13	3.57	23	0	0
14	3.26	25	0	1
15	3.53	26	0	0
16	2.74	19	0	0
17	2.75	25	0	0
18	2.83	19	0	0
19	3.12	23	1	0
20	3.16	25	1	1
21	2.06	22	1	0
22	3.62	28	1	1
23	2.89	14	1	0
24	3.51	26	1	0
25	3.54	24	1	1
26	2.83	27	1	1
27	3.39	17	1	1
28	2.67	24	1	0
29	3.65	21	1	1
30	4.00	23	1	1
31	3.10	21	1	0
32	2.39	19	1	1

Figure 1: Raw data.

Group: UNTITLED Workfile: UNTITLED\Untitled

View | Proc | Object | Print | Name | Freeze | Sample | Sheet | Stats | Spec

	GPA	GRADE	PSI	TUCE
Mean	3.117188	0.343750	0.437500	21.93750
Median	3.065000	0.000000	0.000000	22.50000
Maximum	4.000000	1.000000	1.000000	29.00000
Minimum	2.060000	0.000000	0.000000	12.00000
Std. Dev.	0.466713	0.482559	0.504016	3.901509
Skewness	0.122657	0.657952	0.251976	-0.525110
Kurtosis	2.570068	1.432900	1.063492	3.048305
Jarque-Bera	0.326695	5.583204	5.338708	1.473728
Probability	0.849296	0.061323	0.069297	0.478612
Sum	99.75000	11.00000	14.00000	702.0000
Sum Sq. Dev.	6.752447	7.218750	7.875000	471.8750
Observations	32	32	32	32

Figure 2: Descriptive statistics.

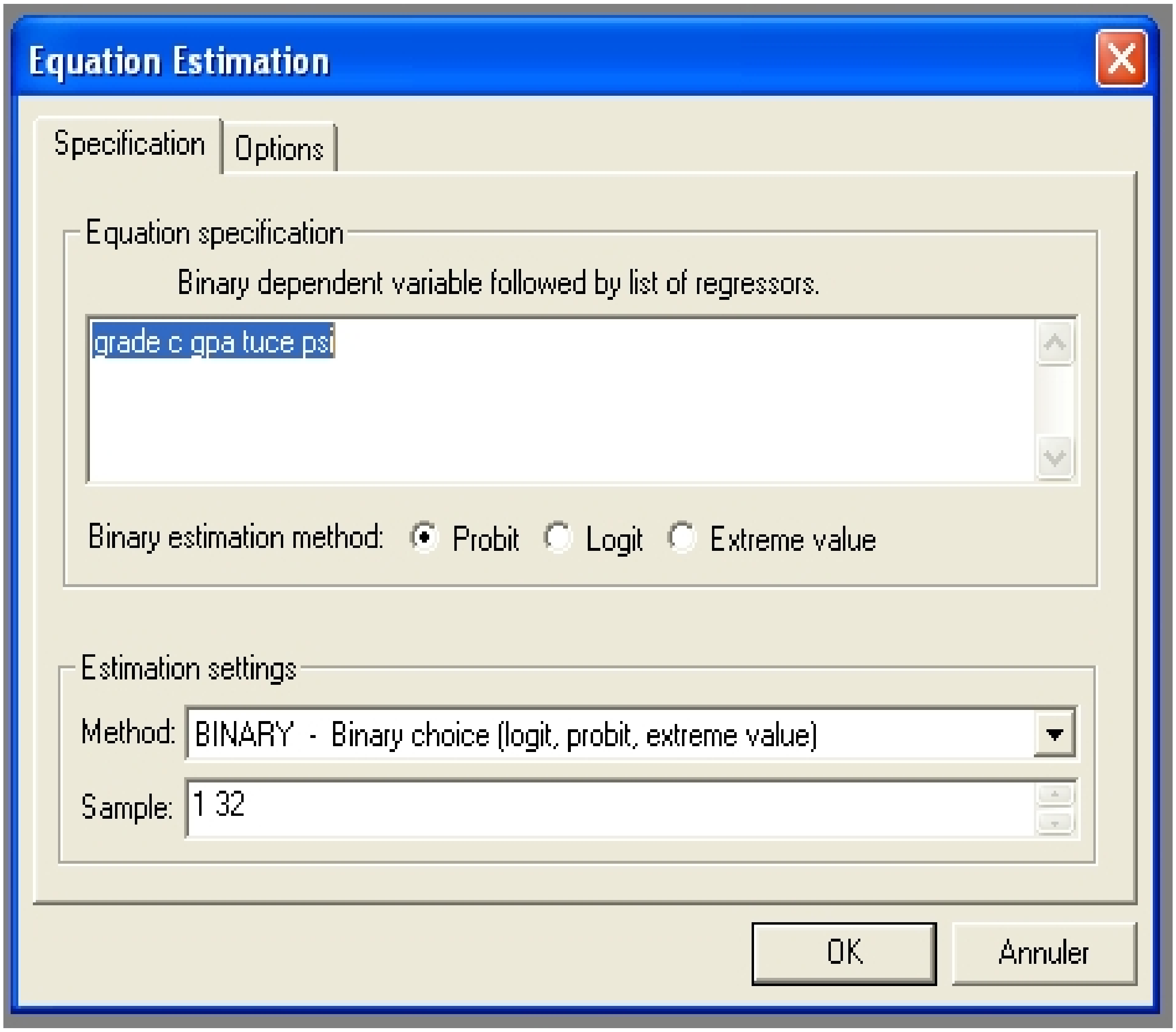


Figure 3: Estimate a probit model of grade on constant, gpa, tuce and psi.

Equation: PROBIT_EQN Workfile: UNTITLED\Untitled

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: GRADE
Method: ML - Binary Probit (Quadratic hill climbing)
Date: 08/25/07 Time: 18:34
Sample: 1 32
Included observations: 32
Convergence achieved after 5 iterations
Covariance matrix computed using second derivatives

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	-7.452320	2.542472	-2.931131	0.0034
GPA	1.625810	0.693882	2.343063	0.0191
TUCE	0.051729	0.083890	0.616626	0.5375
PSI	1.426332	0.595038	2.397045	0.0165

Mean dependent var	0.343750	S.D. dependent var	0.482559
S.E. of regression	0.386128	Akaike info criterion	1.051175
Sum squared resid	4.174660	Schwarz criterion	1.234392
Log likelihood	-12.81880	Hannan-Quinn criter.	1.111906
Restr. log likelihood	-20.59173	Avg. log likelihood	-0.400588
LR statistic (3 df)	15.54585	McFadden R-squared	0.377478
Probability(LR stat)	0.001405		

Obs with Dep=0	21	Total obs	32
Obs with Dep=1	11		

Figure 4: Estimated probit model of grade on constant, gpa, tuce and psi.

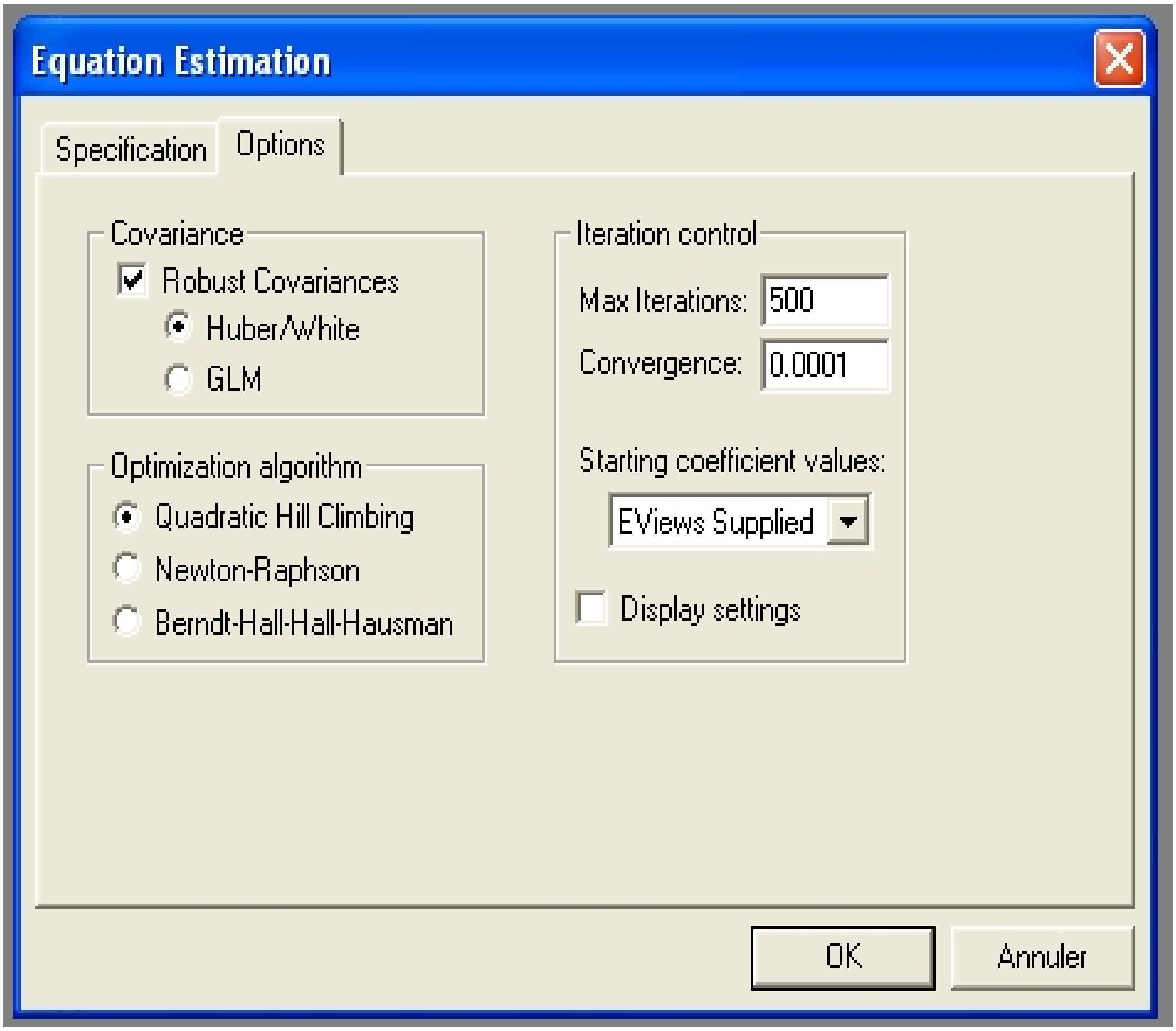


Figure 5: Estimate a probit model of grade on constant, gpa, tuce and psi, using Huber-White robust covariances (these give the asymptotic covariance matrix $\widehat{\text{AVar}}(\widehat{\beta})$).

Equation: PROBIT_EQN_ROBUST Workfile: UNTITLED\U...

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: GRADE
Method: ML - Binary Probit (Quadratic hill climbing)
Date: 11/19/07 Time: 15:34
Sample: 1 32
Included observations: 32
Convergence achieved after 5 iterations
QML (Huber/White) standard errors & covariance

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	-7.452320	2.544271	-2.929059	0.0034
GPA	1.625810	0.651510	2.495448	0.0126
TUCE	0.051729	0.069133	0.748256	0.4543
PSI	1.426332	0.532765	2.677224	0.0074

Mean dependent var	0.343750	S.D. dependent var	0.482559
S.E. of regression	0.386128	Akaike info criterion	1.051175
Sum squared resid	4.174660	Schwarz criterion	1.234392
Log likelihood	-12.81880	Hannan-Quinn criter.	1.111906
Restr. log likelihood	-20.59173	Avg. log likelihood	-0.400588
LR statistic (3 df)	15.54585	McFadden R-squared	0.377478
Probability(LR stat)	0.001405		

Obs with Dep=0	21	Total obs	32
Obs with Dep=1	11		

Figure 6: Estimated probit model of grade on constant, gpa, tuce and psi, using Huber-White robust covariances (this gives the square roots of the diagonal elements of the asymptotic covariance matrix $\widehat{\text{AVar}}(\widehat{\beta})$) as standard errors).

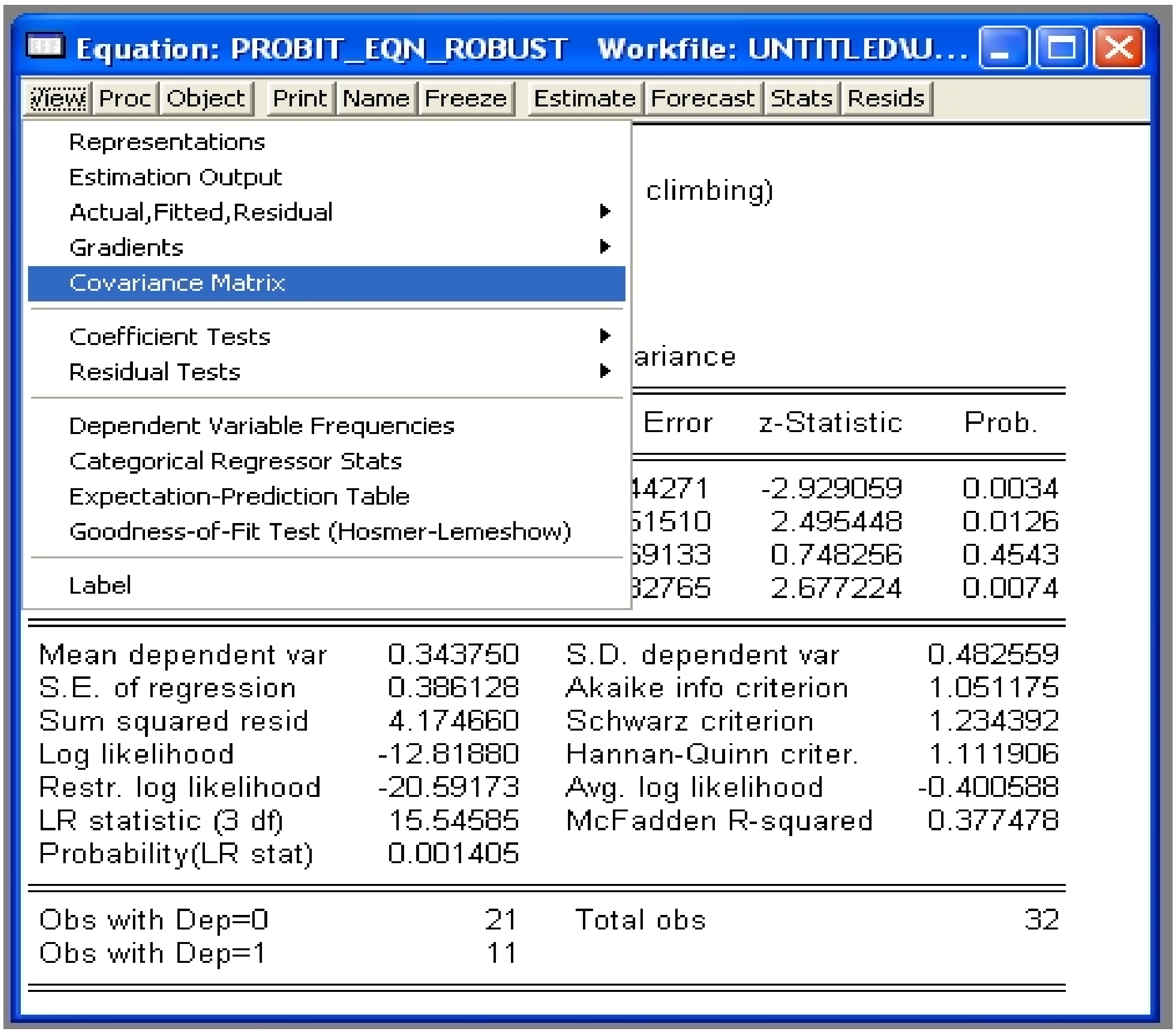
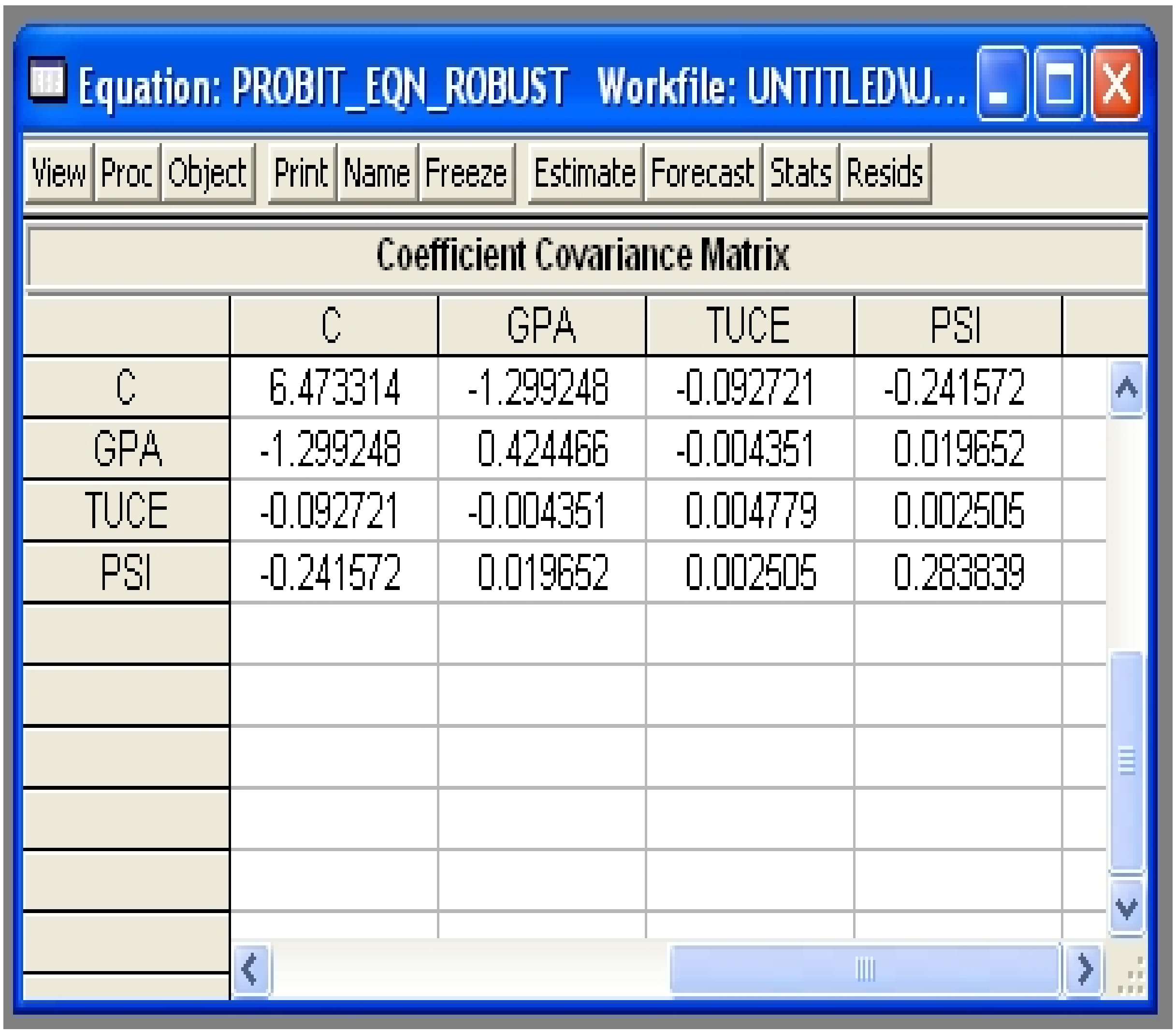


Figure 7: Finding the asymptotic covariance matrix $\widehat{\text{AVar}}(\widehat{\beta})$.



	C	GPA	TUCE	PSI
C	6.473314	-1.299248	-0.092721	-0.241572
GPA	-1.299248	0.424466	-0.004351	0.019652
TUCE	-0.092721	-0.004351	0.004779	0.002505
PSI	-0.241572	0.019652	0.002505	0.283839

Figure 8: The asymptotic covariance matrix $\widehat{\text{AVar}}(\widehat{\beta})$.

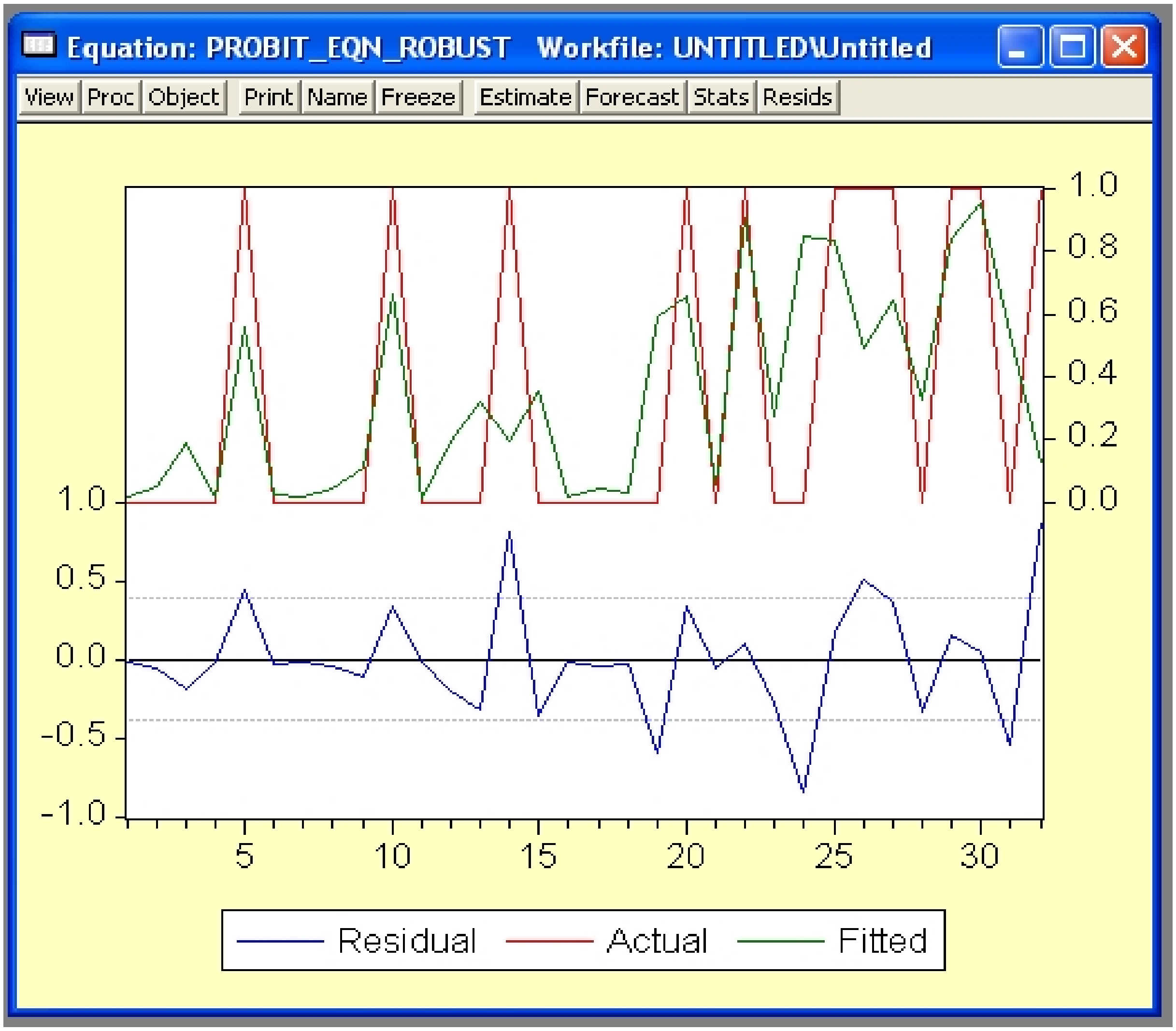
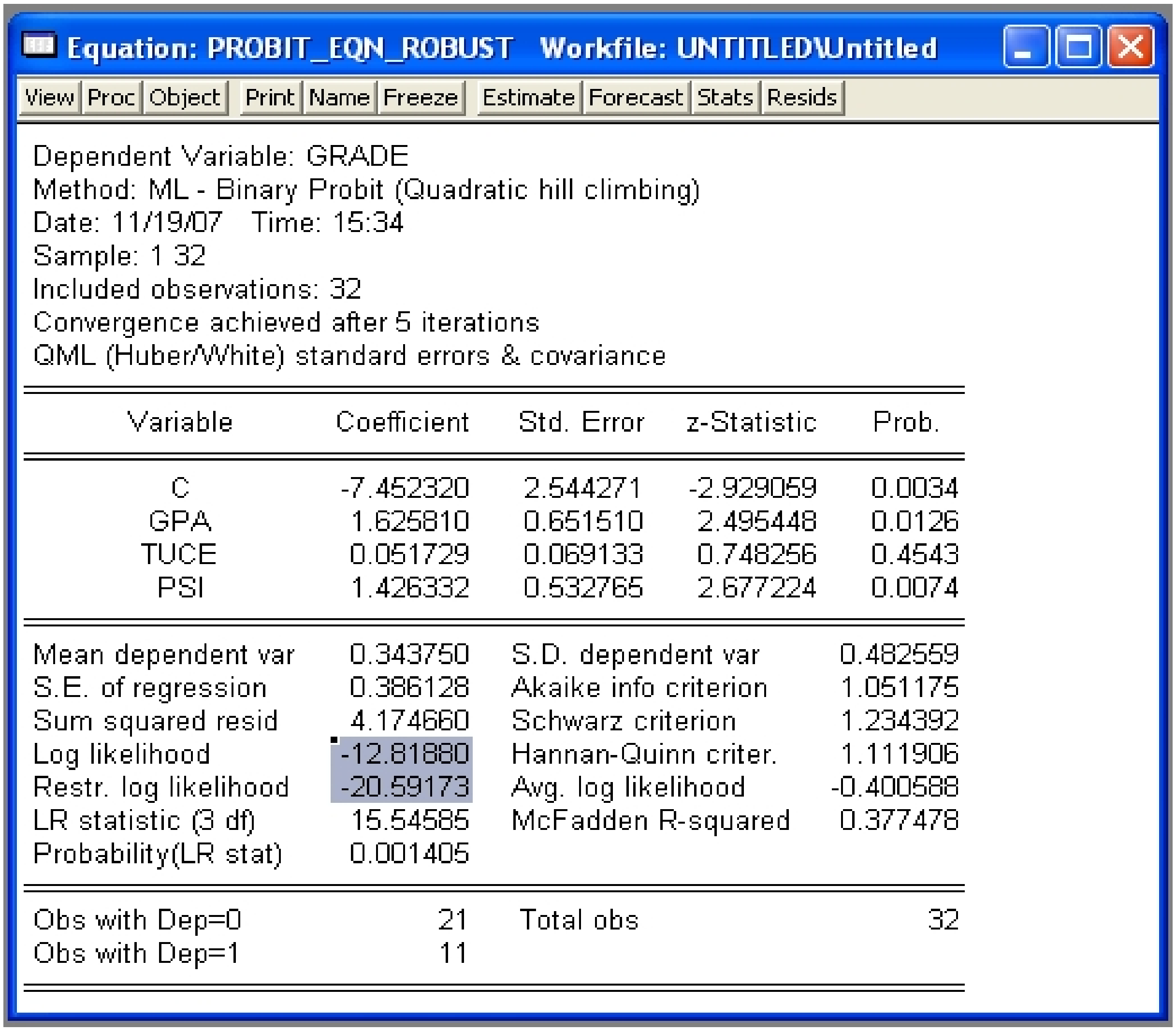


Figure 9: Fitted probabilities and estimated residuals of the probit model.



Equation: PROBIT_EQN_ROBUST Workfile: UNTITLED\Untitled

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: GRADE
Method: ML - Binary Probit (Quadratic hill climbing)
Date: 11/19/07 Time: 15:34
Sample: 1 32
Included observations: 32
Convergence achieved after 5 iterations
QML (Huber/White) standard errors & covariance

Variable	Coefficient	Std. Error	z-Statistic	Prob.
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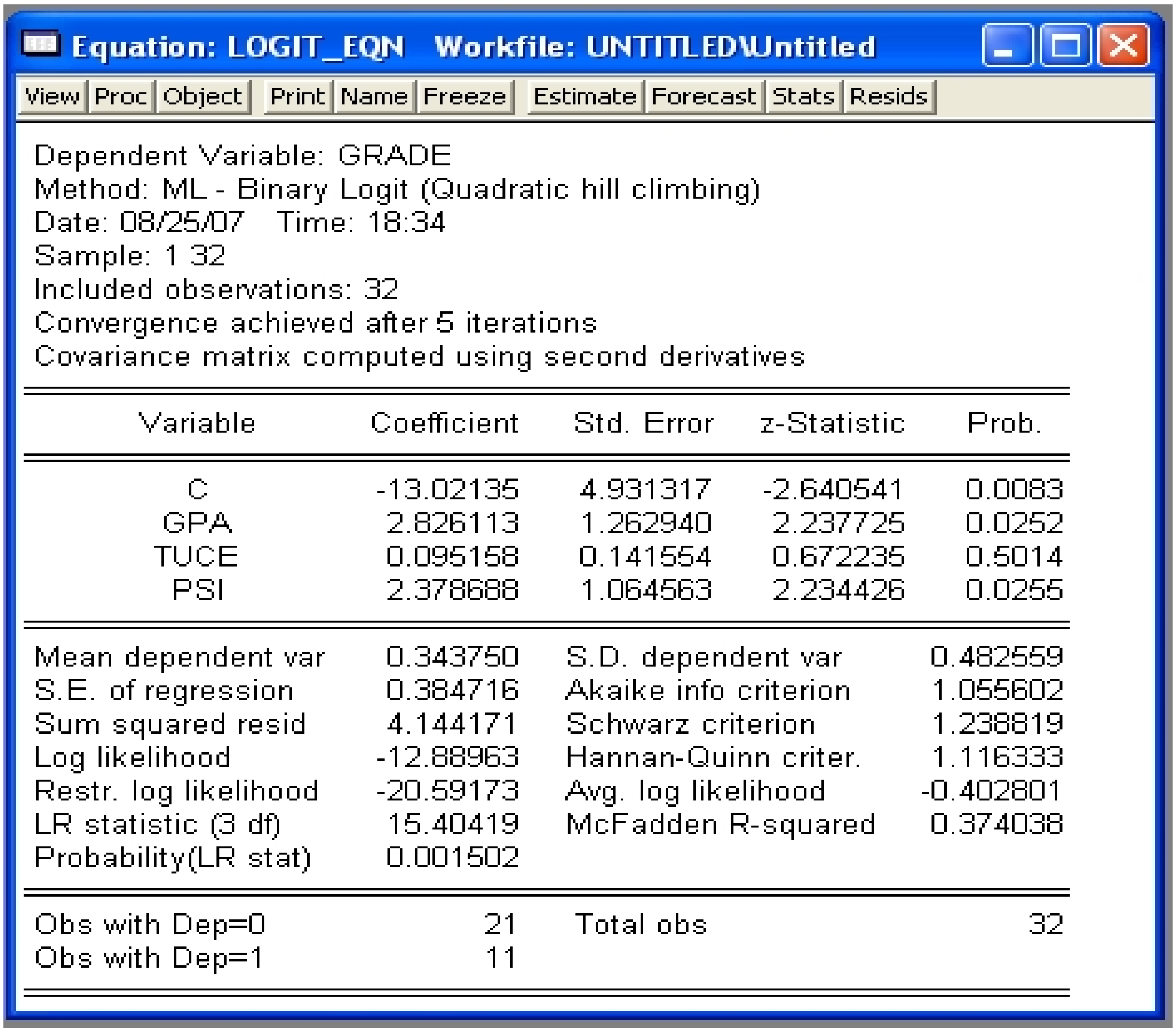
TUCE	0.051729	0.069133	0.748256	0.4543
PSI	1.426332	0.532765	2.677224	0.0074

Mean dependent var	0.343750	S.D. dependent var	0.482559
S.E. of regression	0.386128	Akaike info criterion	1.051175
Sum squared resid	4.174660	Schwarz criterion	1.234392
Log likelihood	-12.81880	Hannan-Quinn criter.	1.111906
Restr. log likelihood	-20.59173	Avg. log likelihood	-0.400588
LR statistic (3 df)	15.54585	McFadden R-squared	0.377478
Probability(LR stat)	0.001405		

Obs with Dep=0	21	Total obs	32
Obs with Dep=1	11		

Figure 10: Fitted log-likelihood, $\ln \widehat{L}$ (Log likelihood) and fitted log-likelihood under the restriction that all coefficients except constant are zero, $\ln \widehat{L}_0$ (Restr. log likelihood).



Equation: LOGIT_EQN Workfile: UNTITLED\Untitled

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: GRADE
Method: ML - Binary Logit (Quadratic hill climbing)
Date: 08/25/07 Time: 18:34
Sample: 1 32
Included observations: 32
Convergence achieved after 5 iterations
Covariance matrix computed using second derivatives

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	-13.02135	4.931317	-2.640541	0.0083
GPA	2.826113	1.262940	2.237725	0.0252
TUCE	0.095158	0.141554	0.672235	0.5014
PSI	2.378688	1.064563	2.234426	0.0255

Mean dependent var	0.343750	S.D. dependent var	0.482559
S.E. of regression	0.384716	Akaike info criterion	1.055602
Sum squared resid	4.144171	Schwarz criterion	1.238819
Log likelihood	-12.88963	Hannan-Quinn criter.	1.116333
Restr. log likelihood	-20.59173	Avg. log likelihood	-0.402801
LR statistic (3 df)	15.40419	McFadden R-squared	0.374038
Probability(LR stat)	0.001502		

Obs with Dep=0	21	Total obs	32
Obs with Dep=1	11		

Figure 11: Estimated logit model of grade on constant, gpa, tuce and psi.

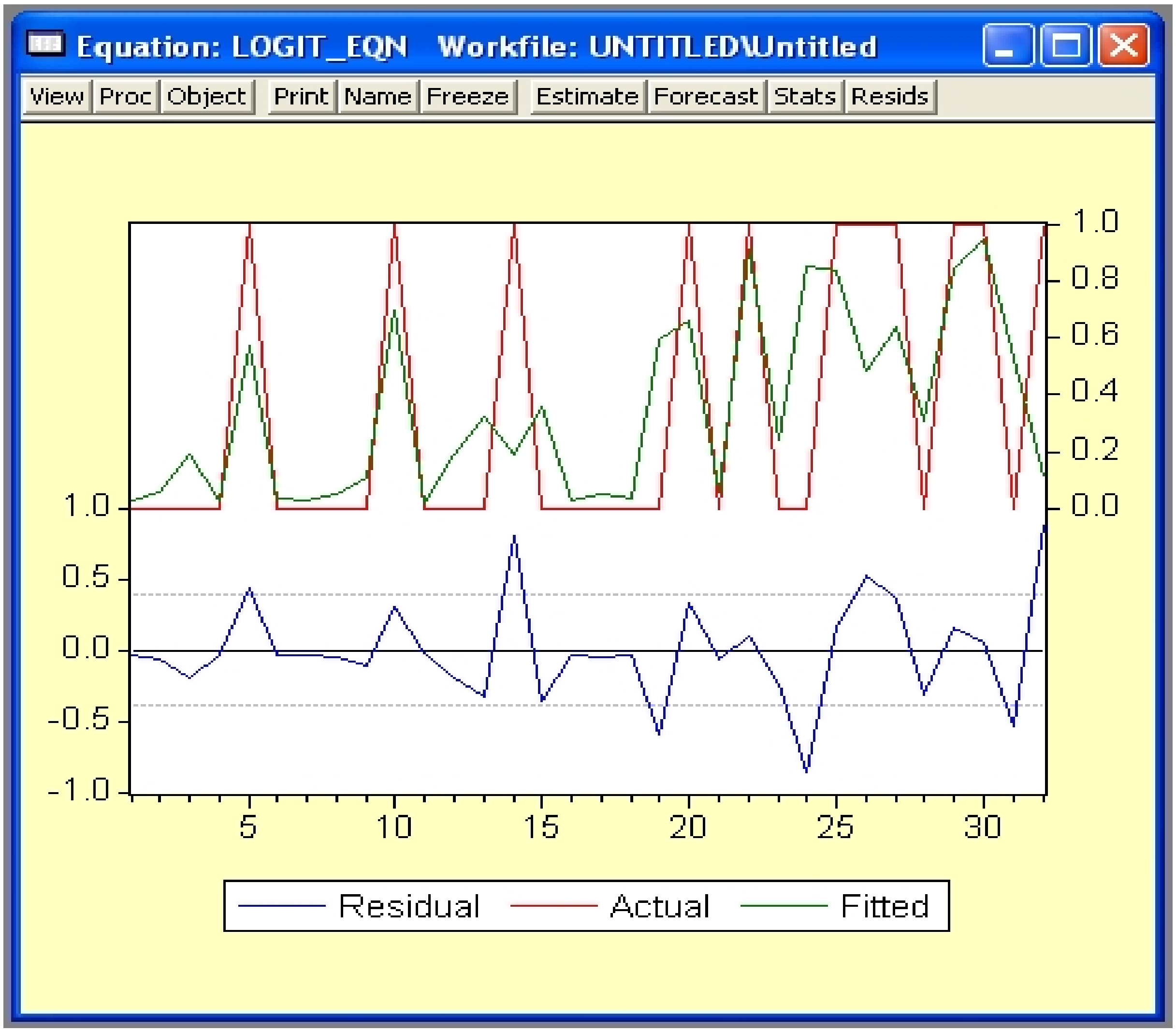


Figure 12: Fitted probabilities and estimated residuals of the logit model.

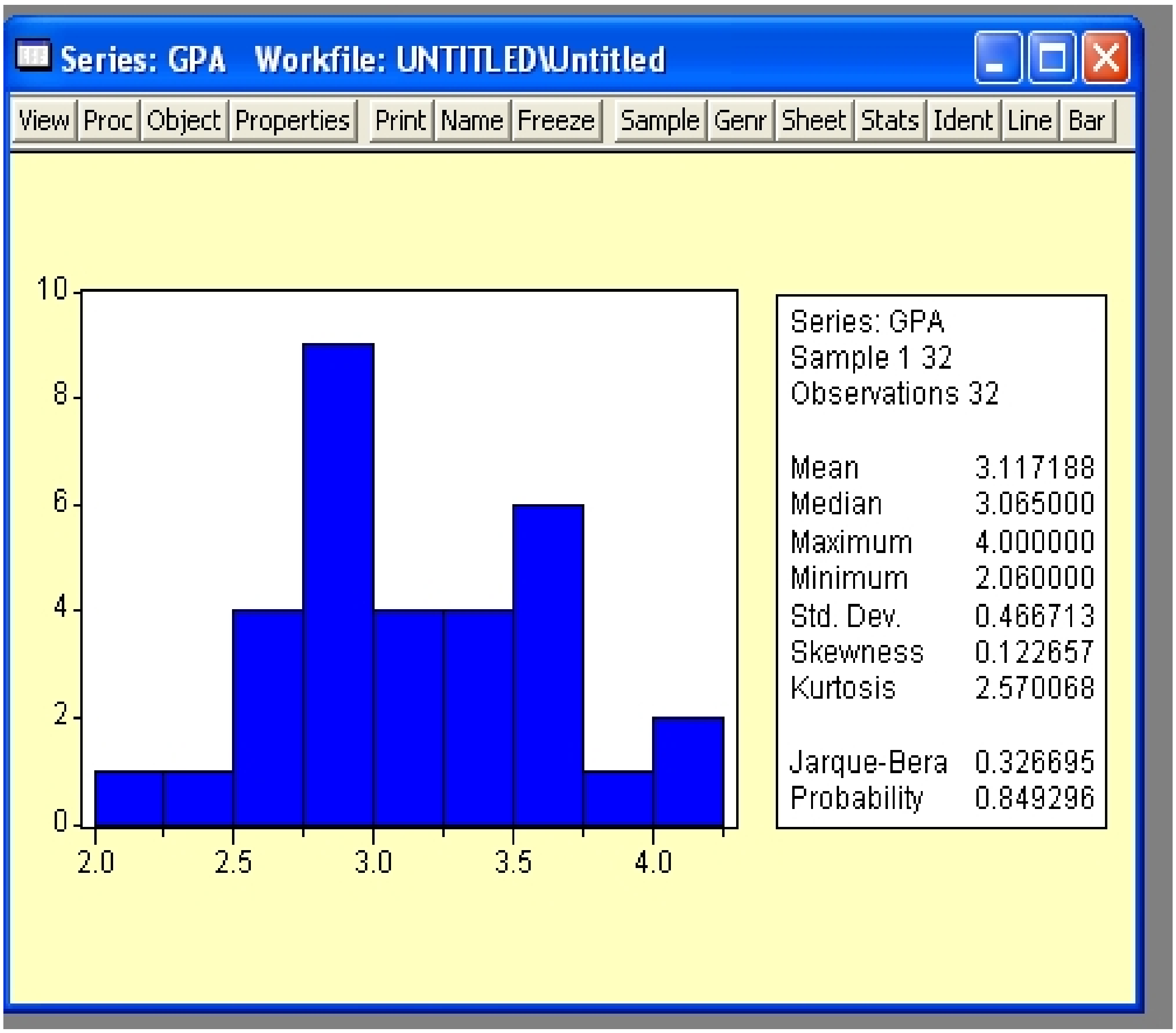


Figure 13: Descriptive statistics on gpa: note the minimum and maximum, in particular.

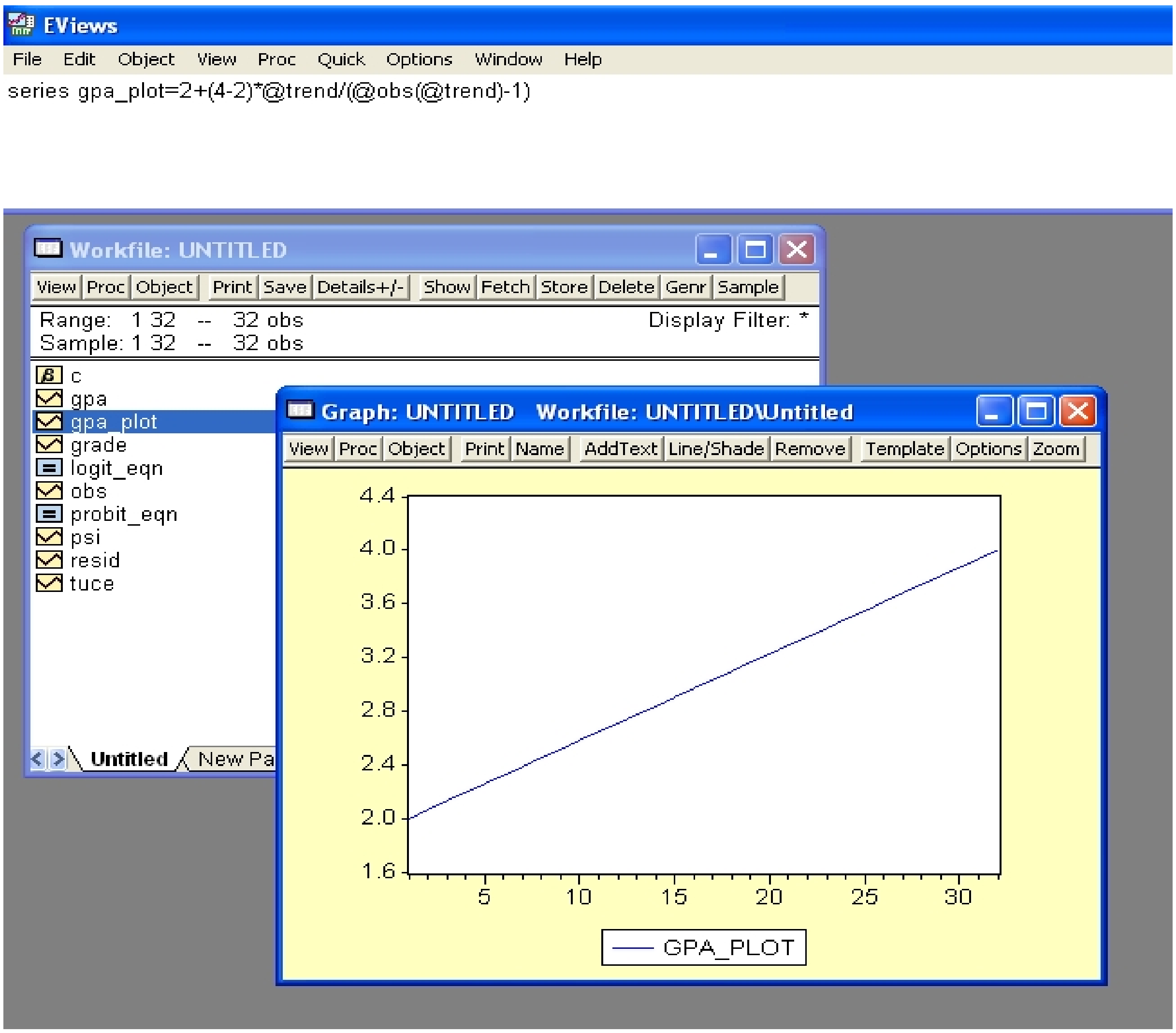


Figure 14: Create an evenly-spaced series on the interval [2, 4].

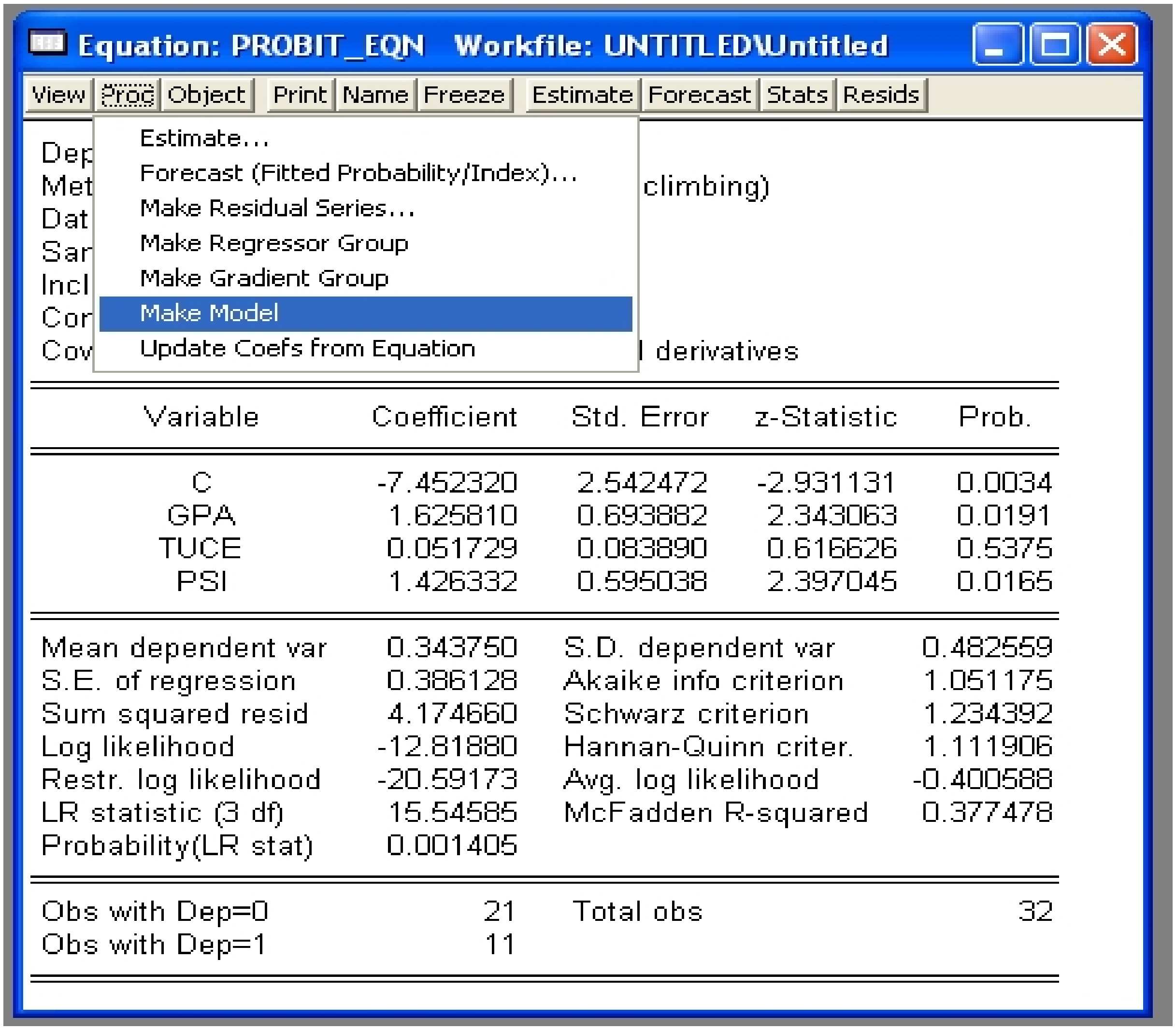


Figure 15: From “probit_eqn”, choose “Make Model”.

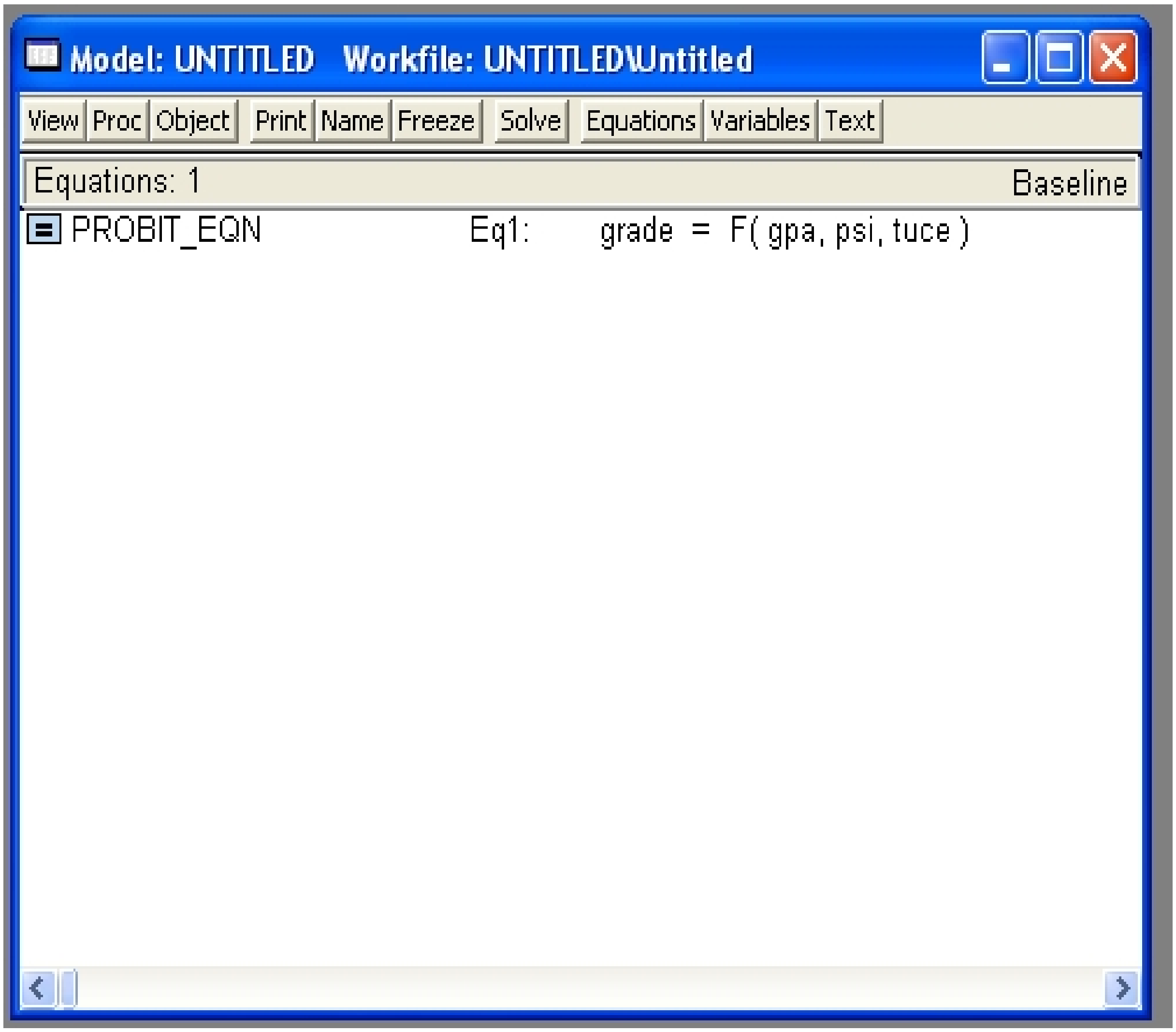


Figure 16: Structure underlying “probit_eqn”.

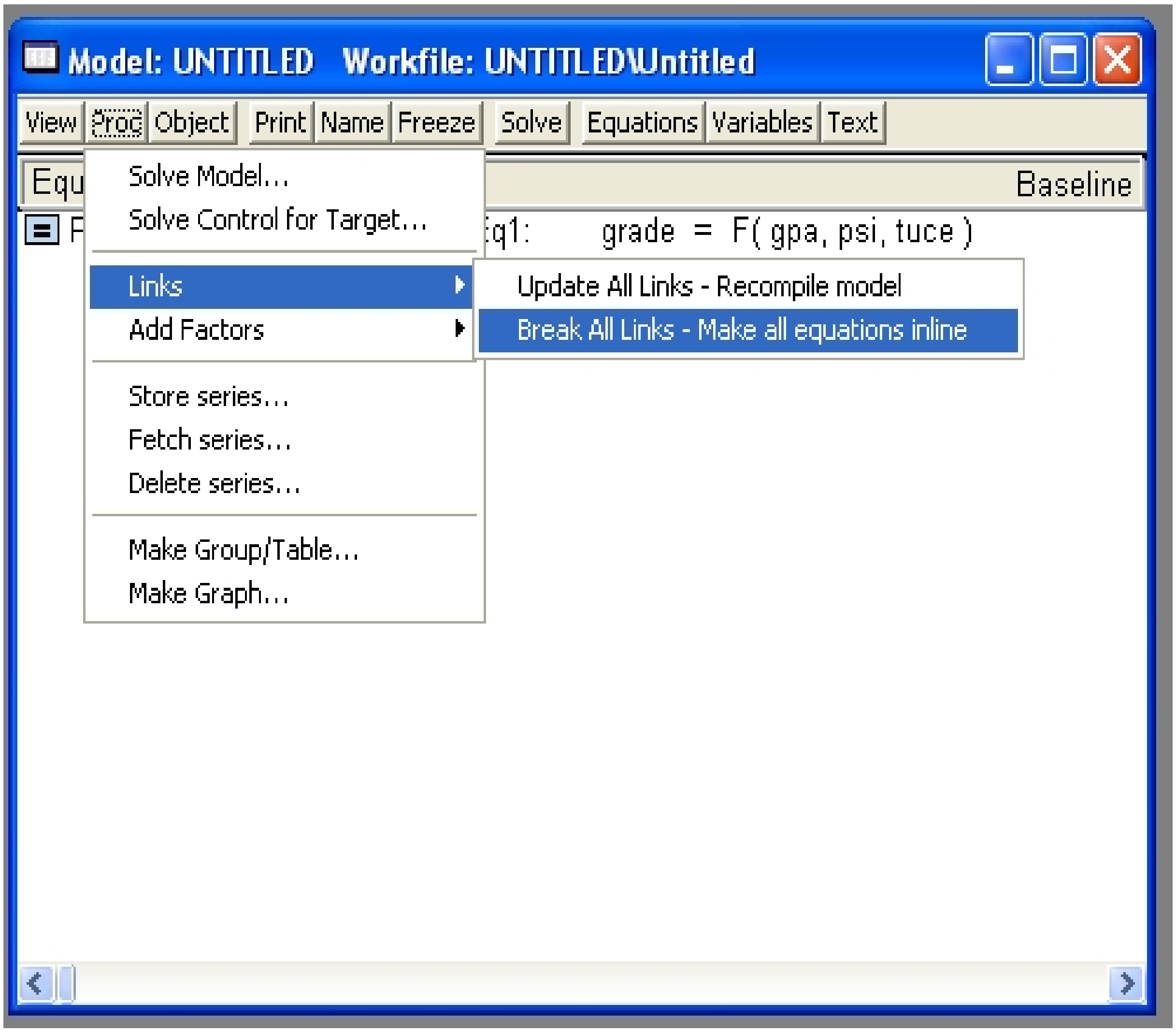


Figure 17: Choose “Break All Links”.

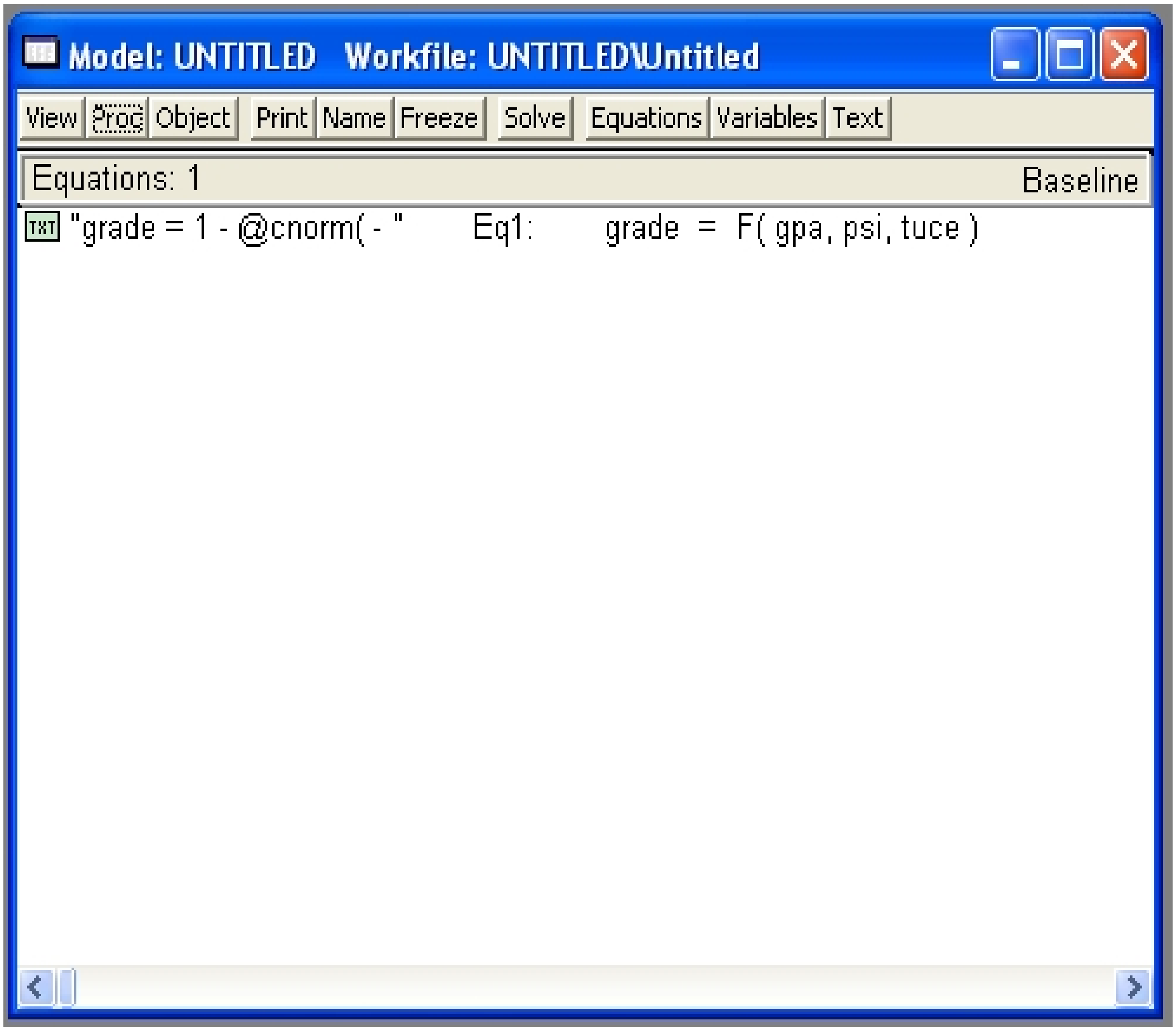


Figure 18: Further structure underlying "probit_eqn".

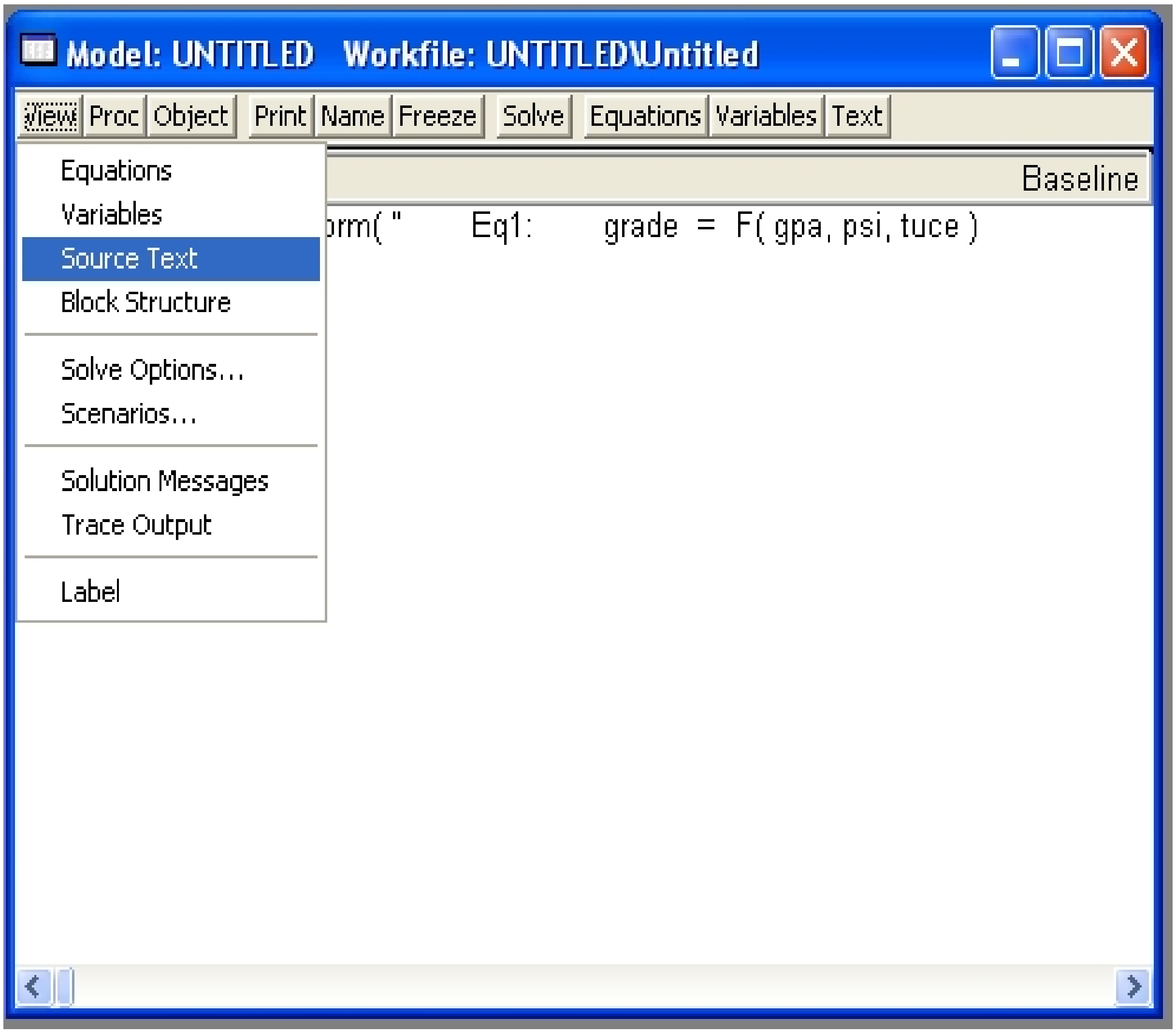


Figure 19: Choose “Source Text”.

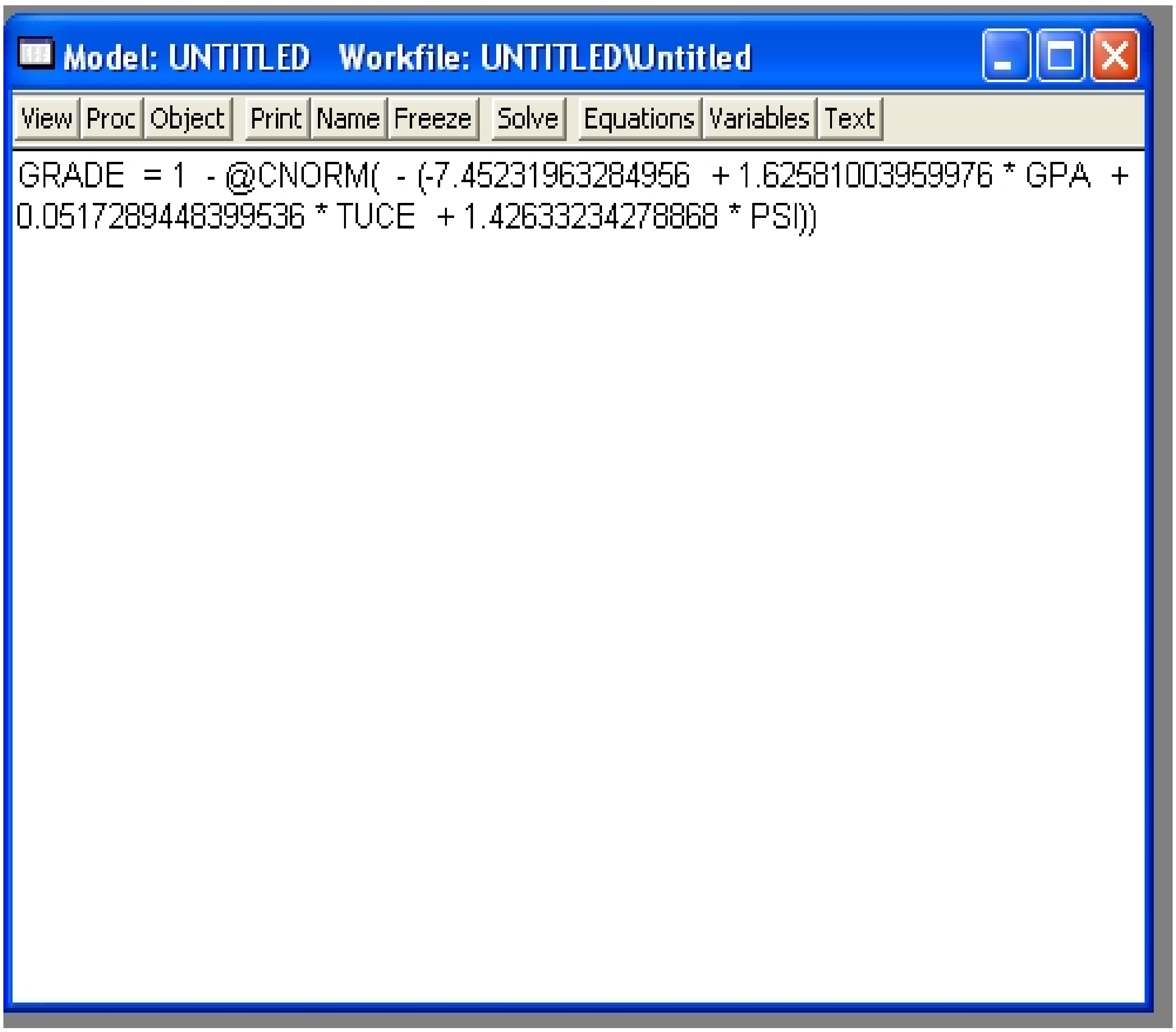


Figure 20: The estimated equation “probit_eqn”.

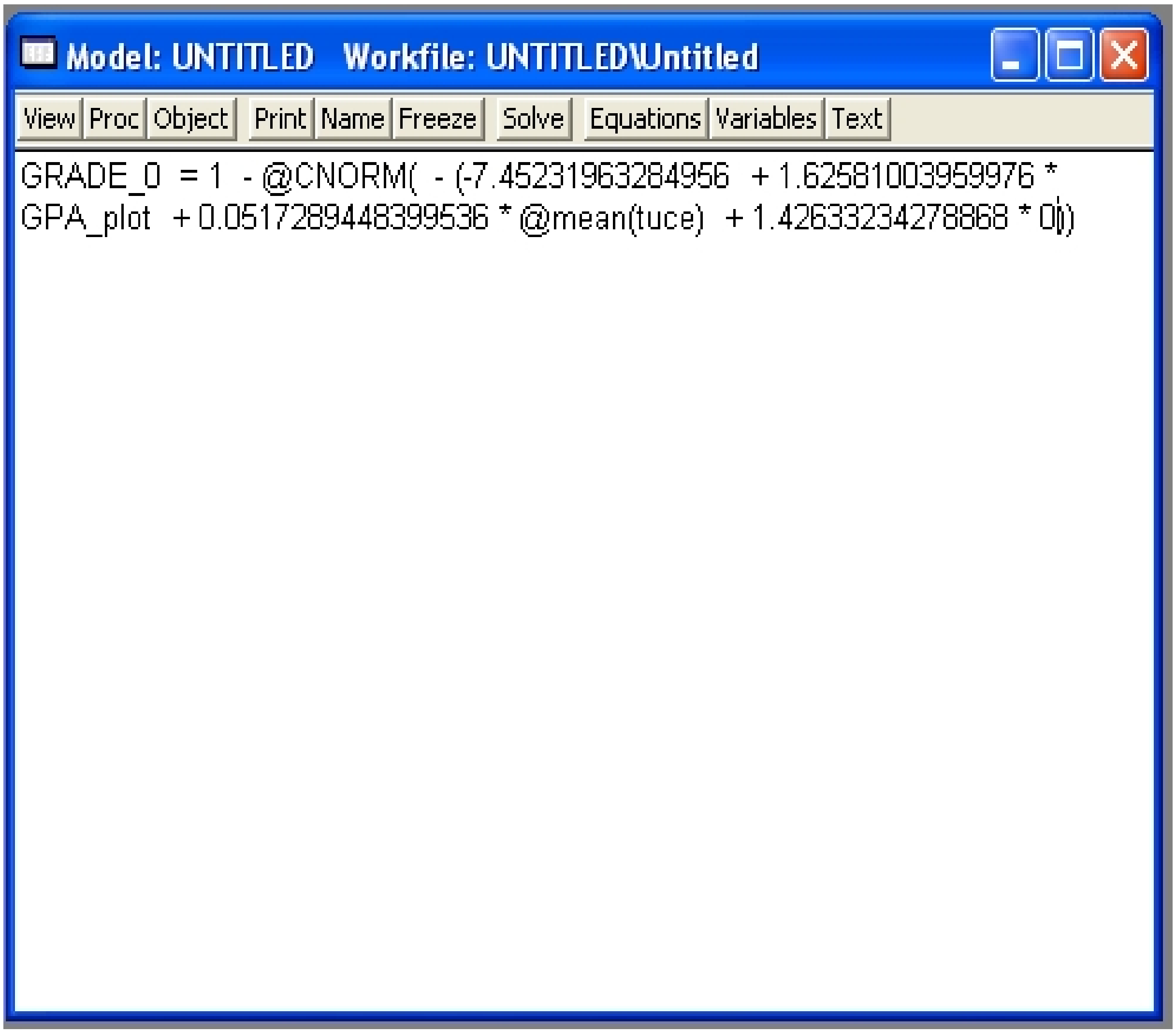


Figure 21: Modify the estimated equation: rename as “GRADE_0”, replace tuce by its sample mean, replace GPA by GPA_plot, and set psi = 0 (no exposure to new method).

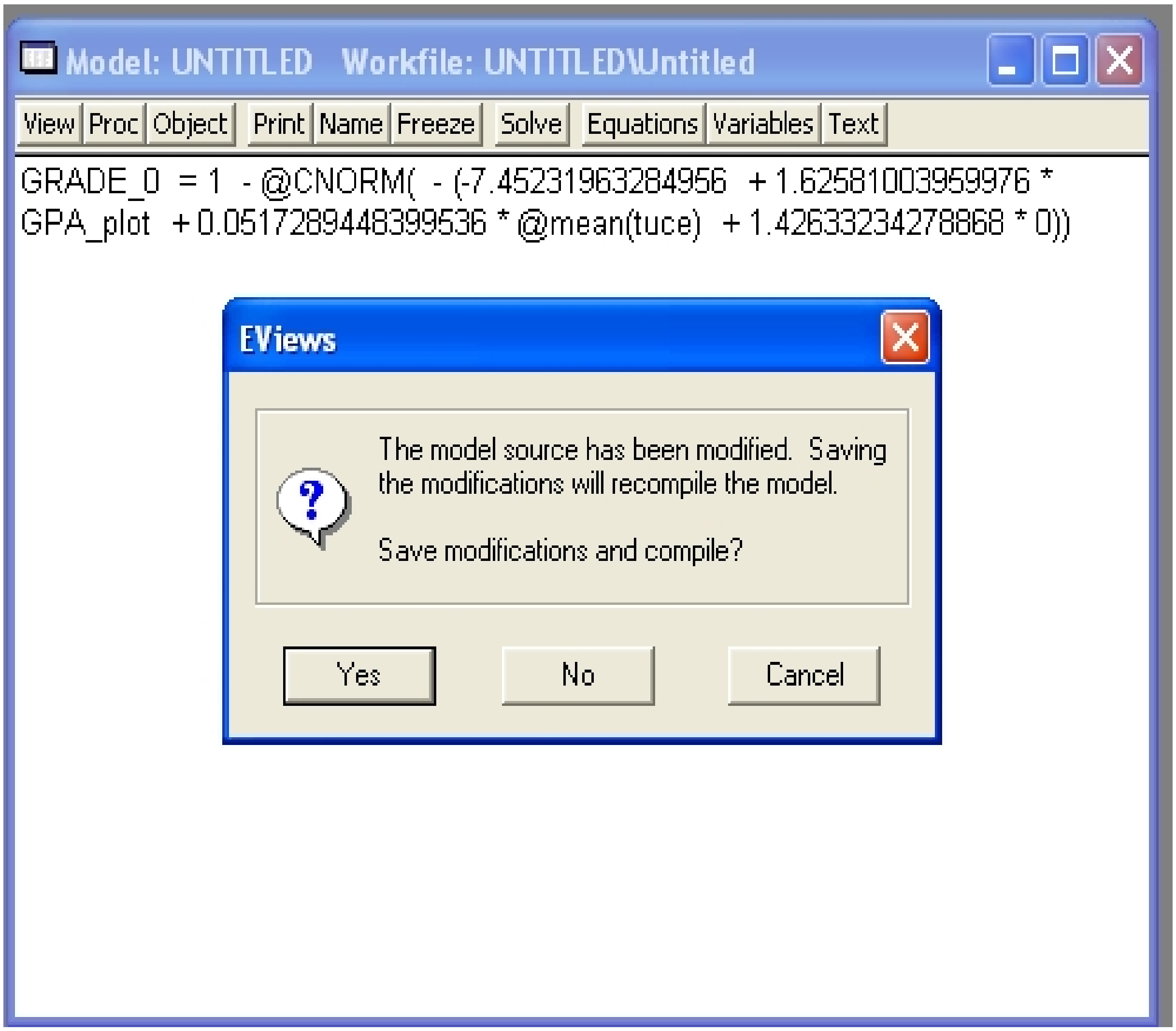


Figure 22: Save modifications to “GRADE_0”.

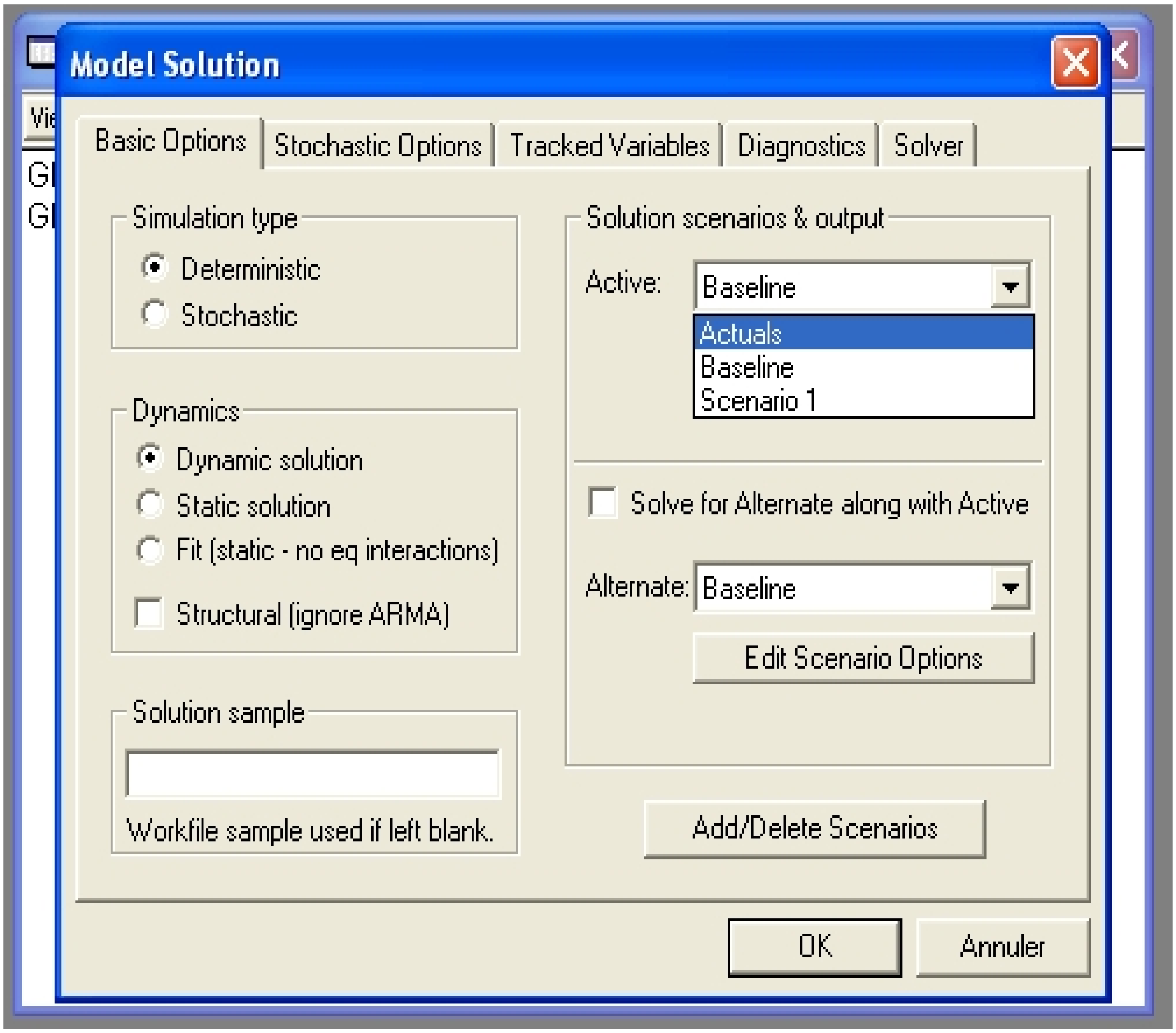


Figure 23: Select "Actuals".

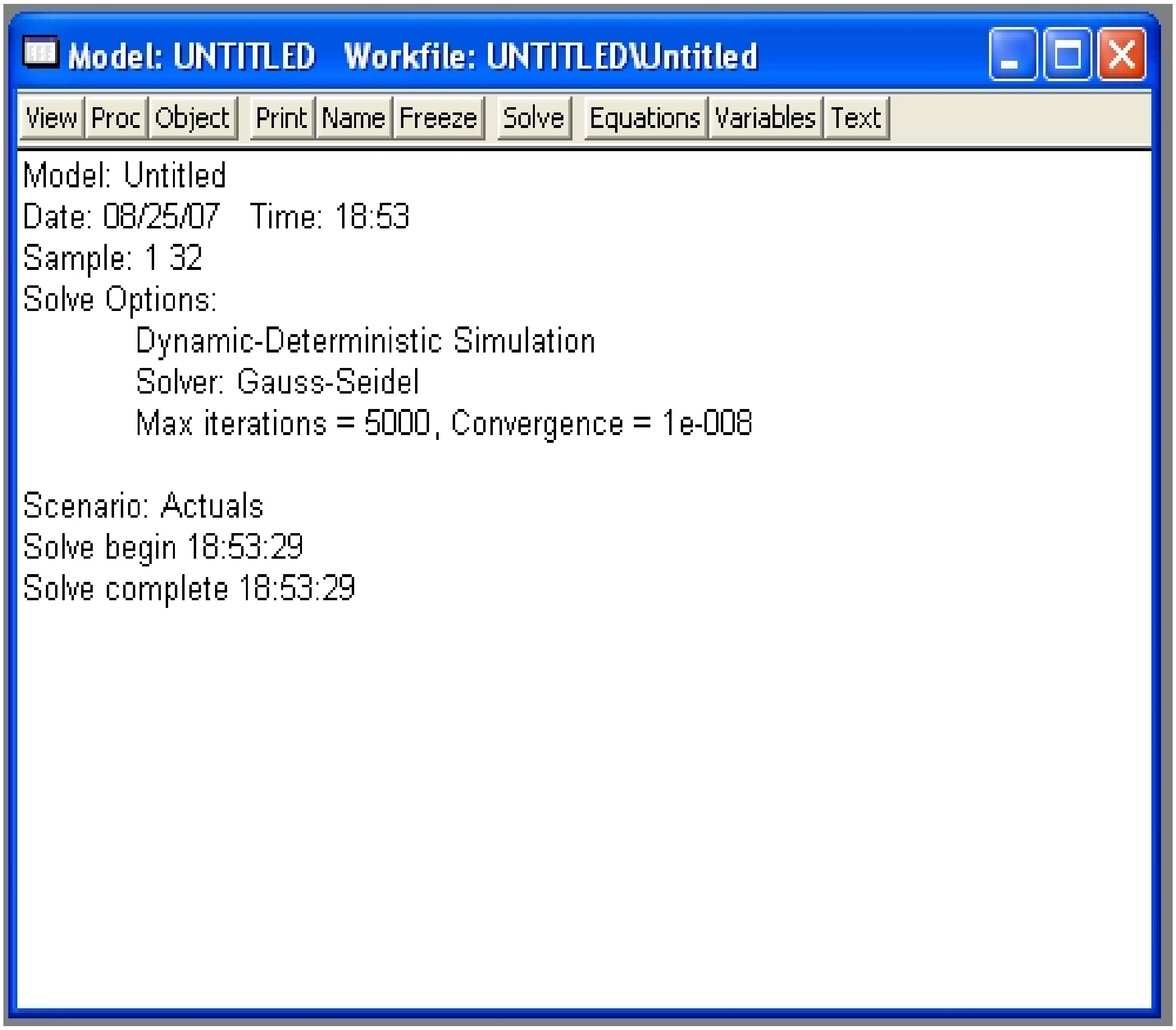


Figure 24: Solve new model "GRADE_0".

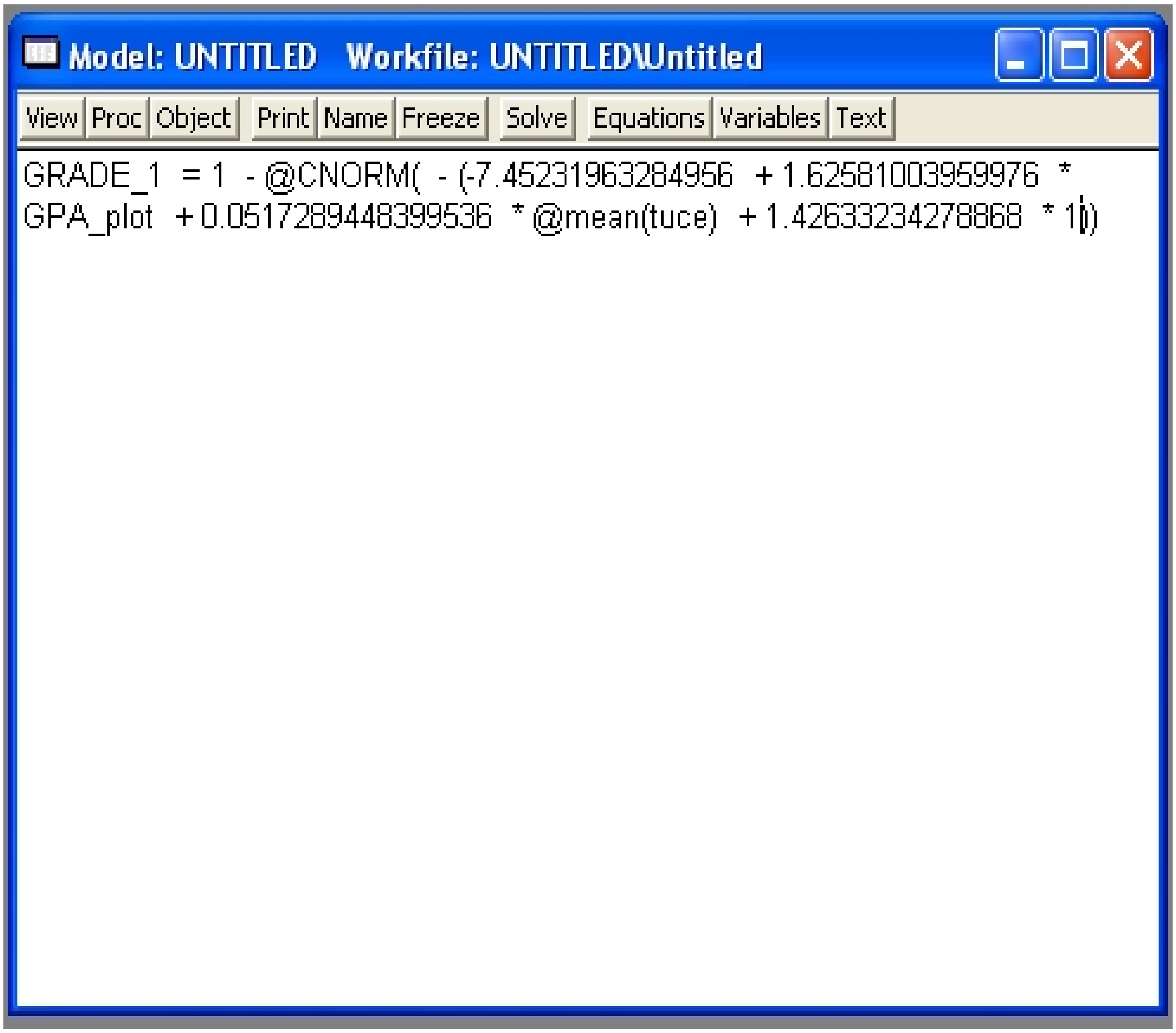


Figure 25: Modify the estimated equation: rename as 'GRADE_1', replace tuce by its sample mean, replace GPA by GPA_plot, and set psi = 1 (exposure to new method).

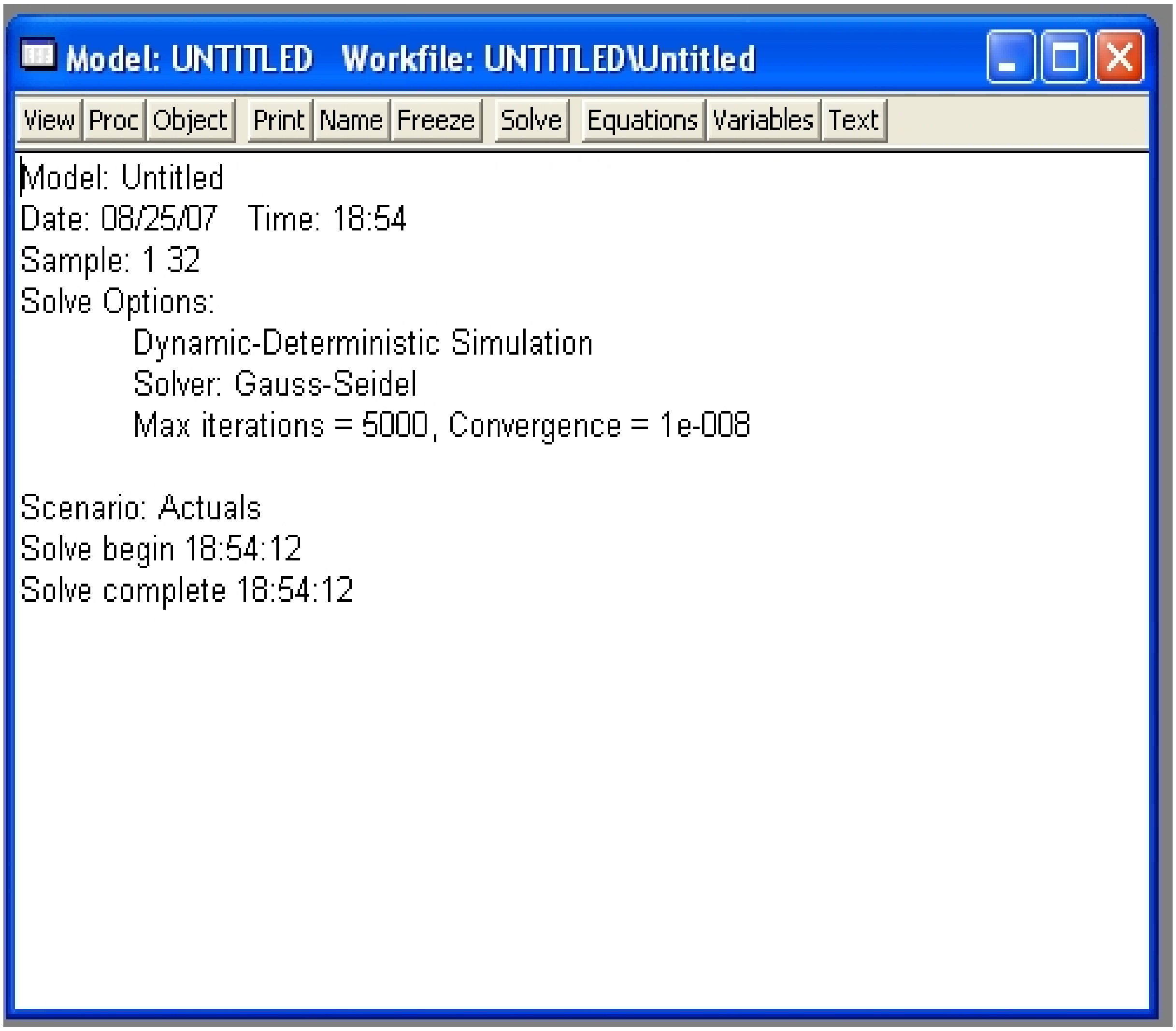


Figure 26: Save modifications to "GRADE_1", select "Actuals", and solve the new model "GRADE_1".

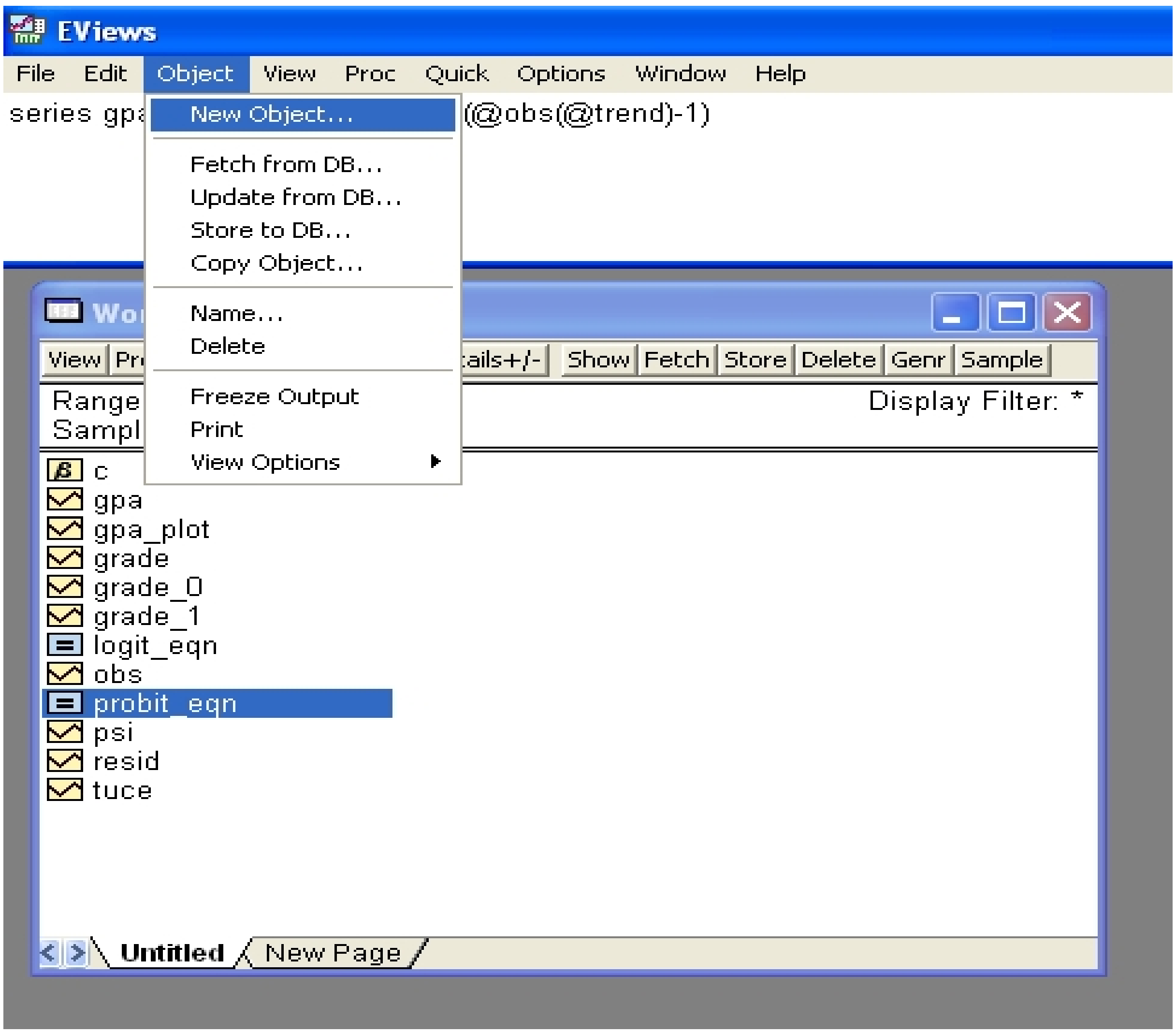


Figure 27: Select "probit_eqn" and "New Object".

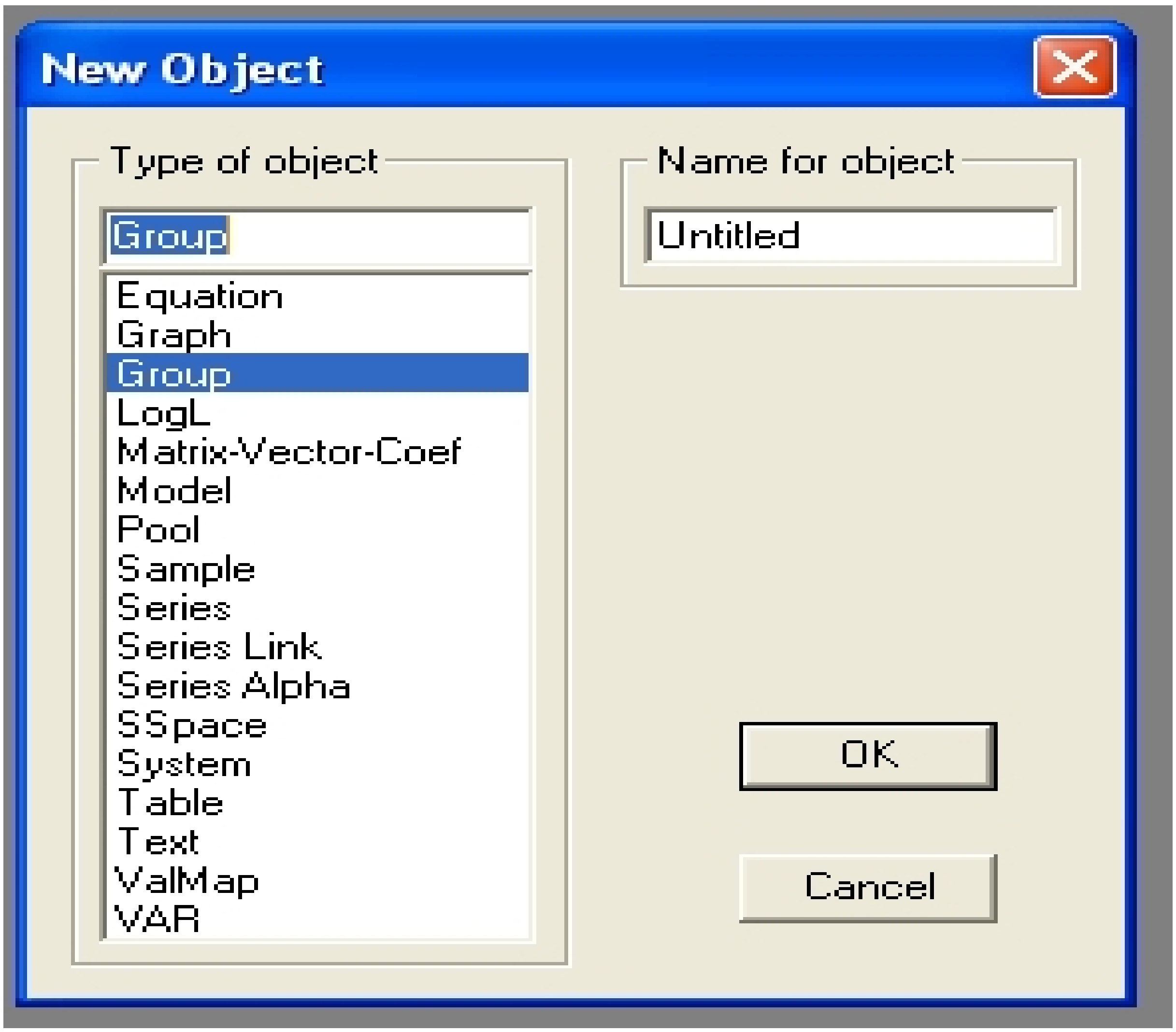


Figure 28: Select “Group”.

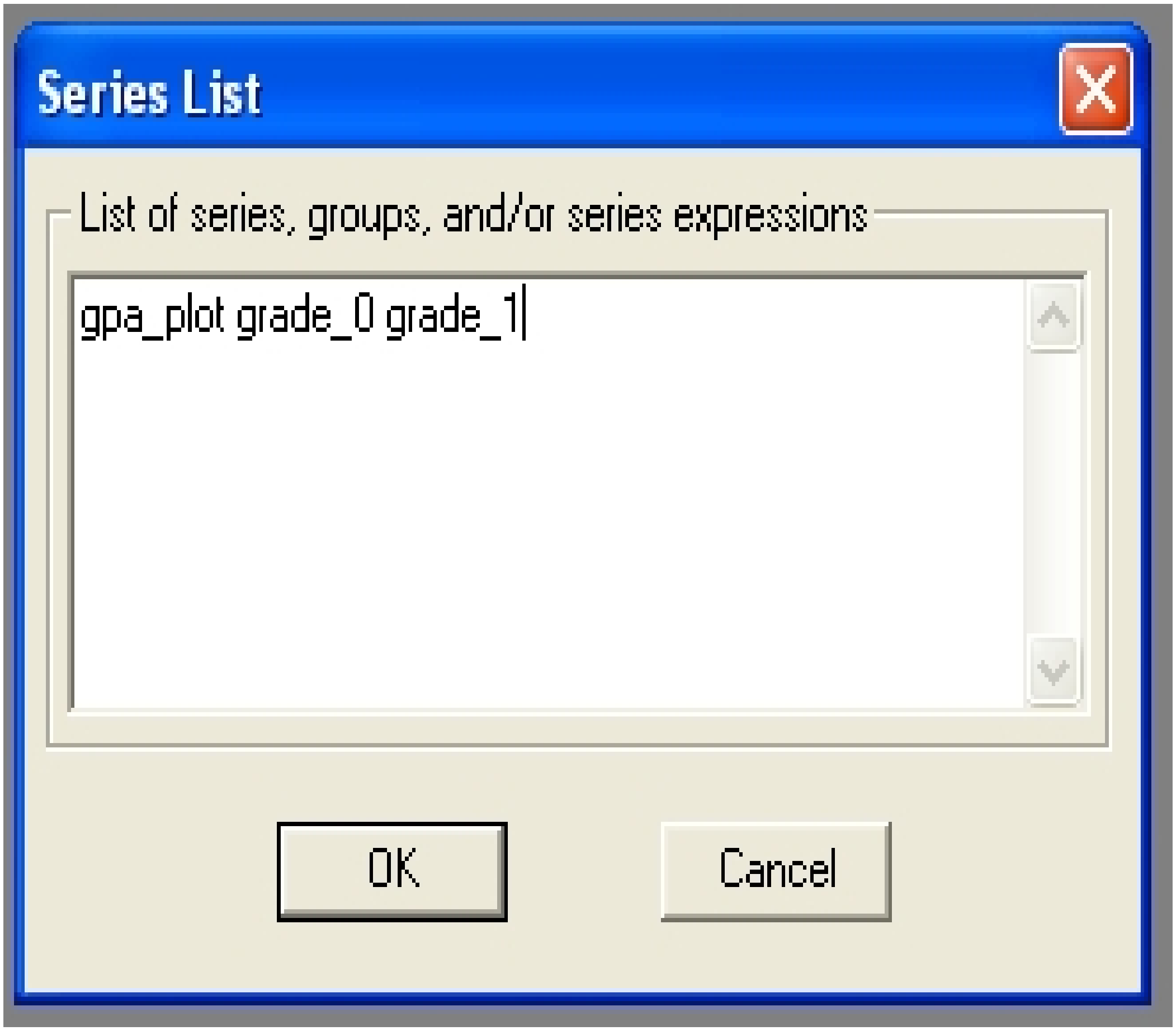
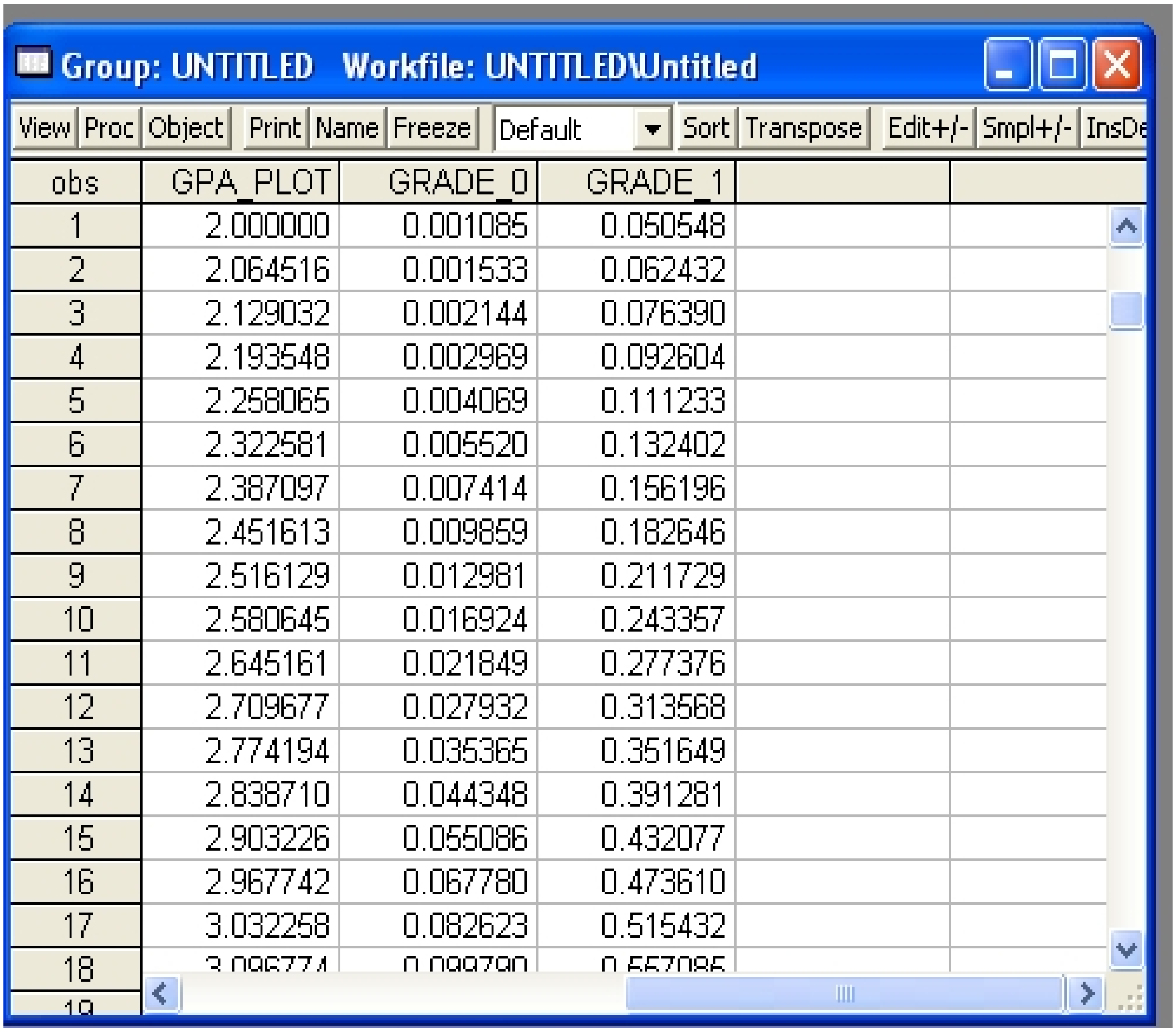


Figure 29: Create group.



obs	GPA_PLOT	GRADE_0	GRADE_1
1	2.000000	0.001085	0.050548
2	2.064516	0.001533	0.062432
3	2.129032	0.002144	0.076390
4	2.193548	0.002969	0.092604
5	2.258065	0.004069	0.111233
6	2.322581	0.005520	0.132402
7	2.387097	0.007414	0.156196
8	2.451613	0.009859	0.182646
9	2.516129	0.012981	0.211729
10	2.580645	0.016924	0.243357
11	2.645161	0.021849	0.277376
12	2.709677	0.027932	0.313568
13	2.774194	0.035365	0.351649
14	2.838710	0.044348	0.391281
15	2.903226	0.055086	0.432077
16	2.967742	0.067780	0.473610
17	3.032258	0.082623	0.515432
18	3.096774	0.099790	0.557085

Figure 30: Data in new group.

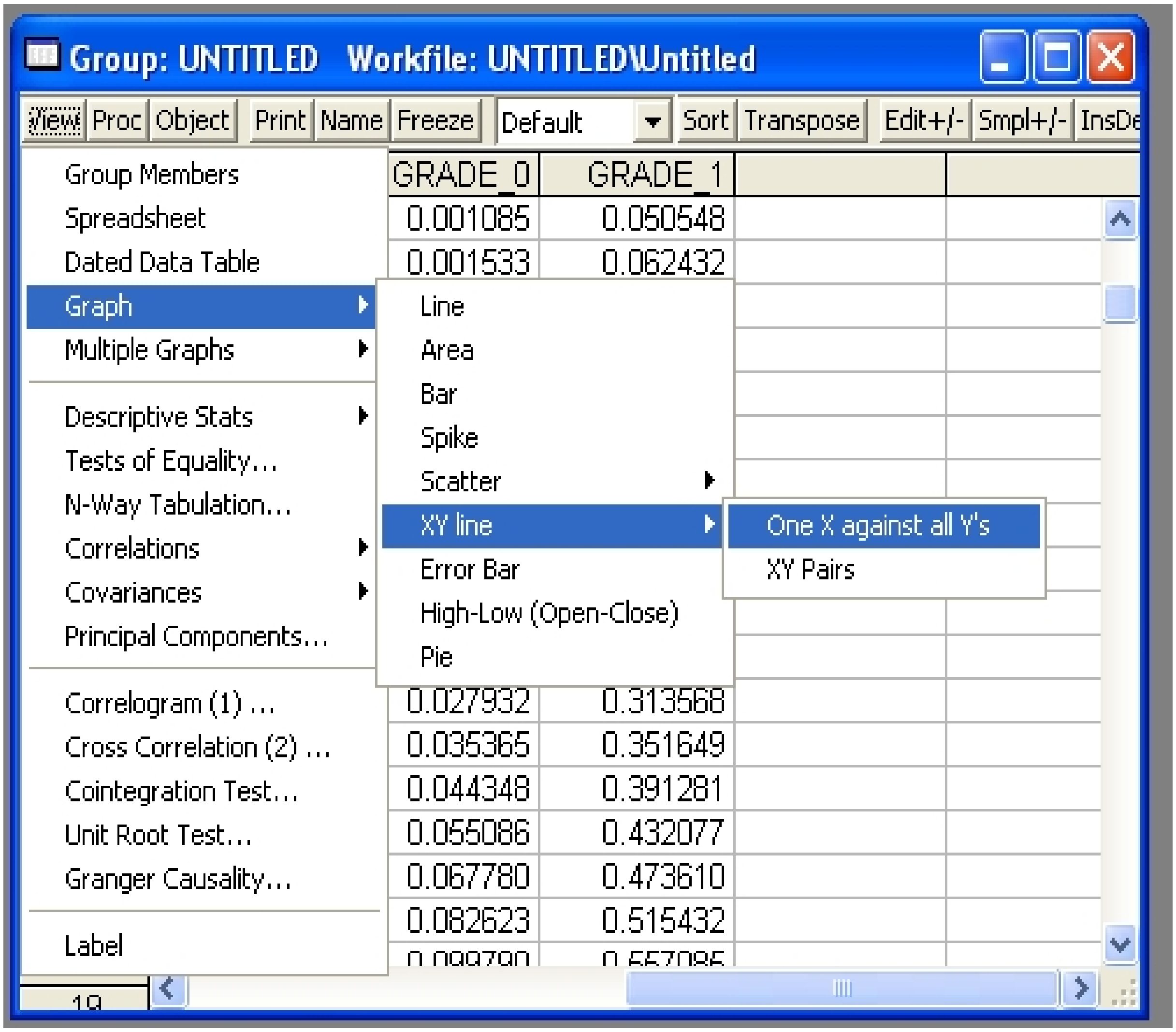


Figure 31: Plot graph based on new group.

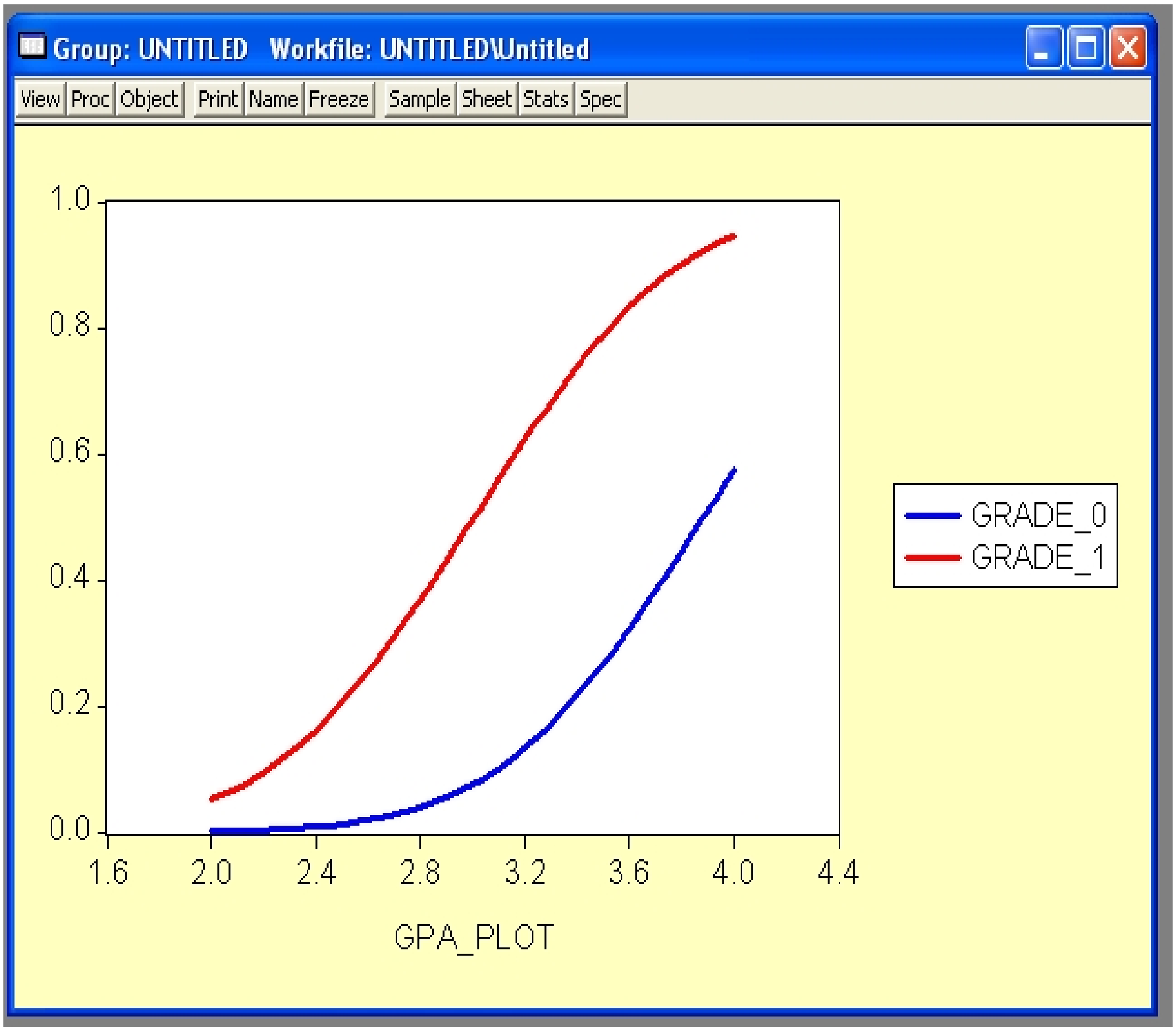


Figure 32: Probability response curves $\widehat{\text{Prob}}(y = 1)$ against gpa, for psi = 0 and psi = 1, for the probit model.

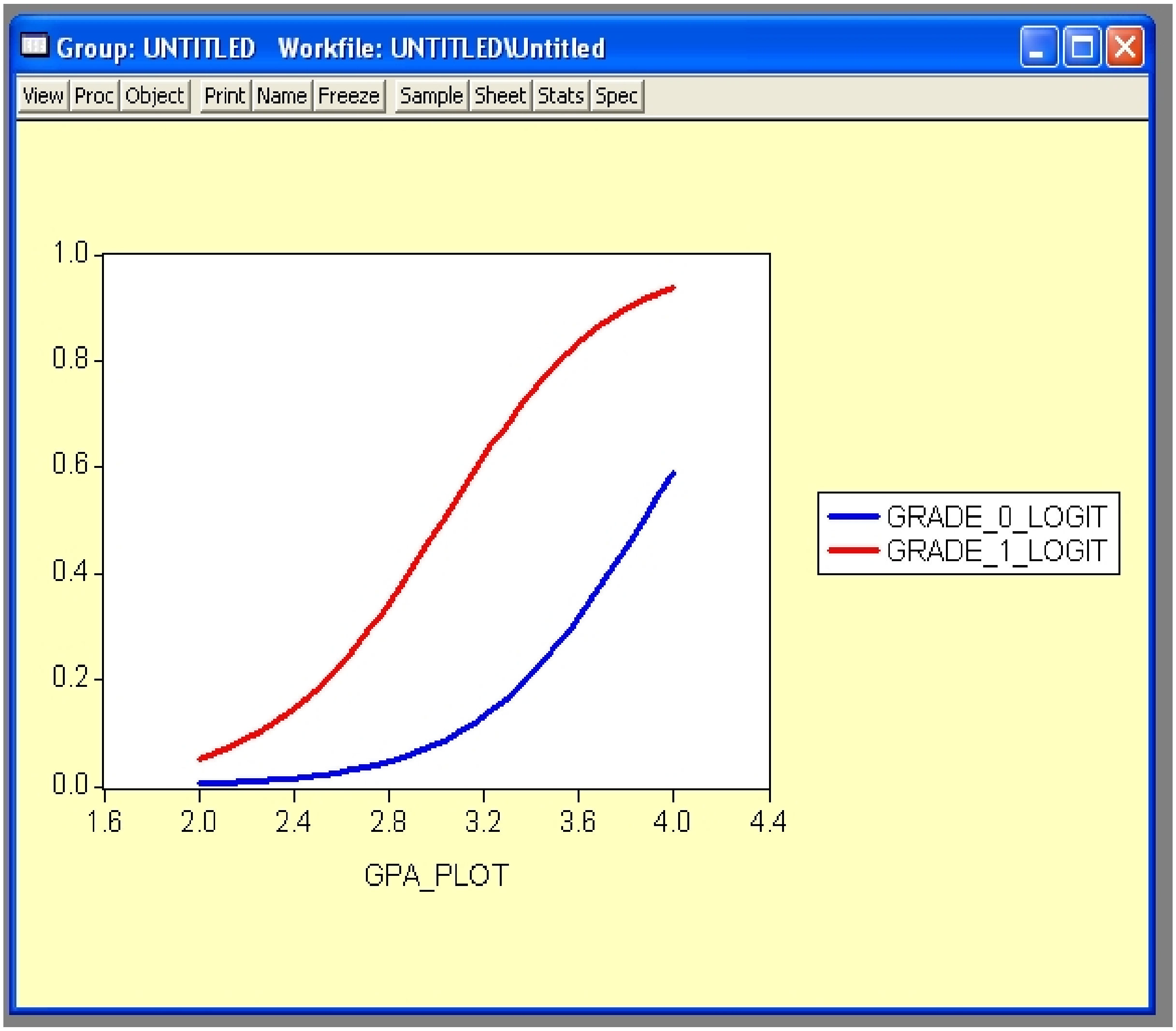


Figure 33: Probability response curves $\widehat{\text{Prob}}(y = 1)$ against gpa, for psi = 0 and psi = 1, for the logit model.

Areas Under the Normal Curve

Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p
0.00	0.5000	0.5000	0.40	0.6554	0.3446	0.80	0.7881	0.2119	1.20	0.8849	0.1151	1.60	0.9452	0.0548
0.01	0.5040	0.4960	0.41	0.6591	0.3409	0.81	0.7910	0.2090	1.21	0.8869	0.1131	1.61	0.9463	0.0537
0.02	0.5080	0.4920	0.42	0.6628	0.3372	0.82	0.7939	0.2061	1.22	0.8888	0.1112	1.62	0.9474	0.0526
0.03	0.5120	0.4880	0.43	0.6664	0.3336	0.83	0.7967	0.2033	1.23	0.8907	0.1093	1.63	0.9484	0.0516
0.04	0.5160	0.4840	0.44	0.6700	0.3300	0.84	0.7995	0.2005	1.24	0.8925	0.1075	1.64	0.9495	0.0505
0.05	0.5199	0.4801	0.45	0.6736	0.3264	0.85	0.8023	0.1977	1.25	0.8944	0.1056	**1.65**	**0.9505**	**0.0495**
0.06	0.5239	0.4761	0.46	0.6772	0.3228	0.86	0.8051	0.1949	1.26	0.8962	0.1038	1.66	0.9515	0.0485
0.07	0.5279	0.4721	0.47	0.6808	0.3192	0.87	0.8078	0.1922	1.27	0.8980	0.1020	1.67	0.9525	0.0475
0.08	0.5319	0.4681	0.48	0.6844	0.3156	0.88	0.8106	0.1894	1.28	0.8997	0.1003	1.68	0.9535	0.0465
0.09	0.5359	0.4641	0.49	0.6879	0.3121	0.89	0.8133	0.1867	1.29	0.9015	0.0985	1.69	0.9545	0.0455
0.10	0.5398	0.4602	0.50	0.6915	0.3085	0.90	0.8159	0.1841	1.30	0.9032	0.0968	1.70	0.9554	0.0446
0.11	0.5438	0.4562	0.51	0.6950	0.3050	0.91	0.8186	0.1814	1.31	0.9049	0.0951	1.71	0.9564	0.0436
0.12	0.5478	0.4522	0.52	0.6985	0.3015	0.92	0.8212	0.1788	1.32	0.9066	0.0934	1.72	0.9573	0.0427
0.13	0.5517	0.4483	0.53	0.7019	0.2981	0.93	0.8238	0.1762	1.33	0.9082	0.0918	1.73	0.9582	0.0418
0.14	0.5557	0.4443	0.54	0.7054	0.2946	0.94	0.8264	0.1736	1.34	0.9099	0.0901	1.74	0.9591	0.0409
0.15	0.5596	0.4404	0.55	0.7088	0.2912	0.95	0.8289	0.1711	1.35	0.9115	0.0885	1.75	0.9599	0.0401
0.16	0.5636	0.4364	0.56	0.7123	0.2877	0.96	0.8315	0.1685	1.36	0.9131	0.0869	1.76	0.9608	0.0392
0.17	0.5675	0.4325	0.57	0.7157	0.2843	0.97	0.8340	0.1660	1.37	0.9147	0.0853	1.77	0.9616	0.0384
0.18	0.5714	0.4286	0.58	0.7190	0.2810	0.98	0.8365	0.1635	1.38	0.9162	0.0838	1.78	0.9625	0.0375
0.19	0.5753	0.4247	0.59	0.7224	0.2776	0.99	0.8389	0.1611	1.39	0.9177	0.0823	1.79	0.9633	0.0367
0.20	0.5793	0.4207	0.60	0.7257	0.2743	1.00	0.8413	0.1587	1.40	0.9192	0.0808	1.80	0.9641	0.0359
0.21	0.5832	0.4168	0.61	0.7291	0.2709	1.01	0.8438	0.1562	1.41	0.9207	0.0793	1.81	0.9649	0.0351
0.22	0.5871	0.4129	0.62	0.7324	0.2676	1.02	0.8461	0.1539	1.42	0.9222	0.0778	1.82	0.9656	0.0344
0.23	0.5910	0.4090	0.63	0.7357	0.2643	1.03	0.8485	0.1515	1.43	0.9236	0.0764	1.83	0.9664	0.0336
0.24	0.5948	0.4052	0.64	0.7389	0.2611	1.04	0.8508	0.1492	1.44	0.9251	0.0749	1.84	0.9671	0.0329
0.25	0.5987	0.4013	0.65	0.7422	0.2578	1.05	0.8531	0.1469	1.45	0.9265	0.0735	1.85	0.9678	0.0322
0.26	0.6026	0.3974	0.66	0.7454	0.2546	1.06	0.8554	0.1446	1.46	0.9279	0.0721	1.86	0.9686	0.0314
0.27	0.6064	0.3936	0.67	0.7486	0.2514	1.07	0.8577	0.1423	1.47	0.9292	0.0708	1.87	0.9693	0.0307
0.28	0.6103	0.3897	0.68	0.7517	0.2483	1.08	0.8599	0.1401	1.48	0.9306	0.0694	1.88	0.9699	0.0301
0.29	0.6141	0.3859	0.69	0.7549	0.2451	1.09	0.8621	0.1379	1.49	0.9319	0.0681	1.89	0.9706	0.0294
0.30	0.6179	0.3821	0.70	0.7580	0.2420	1.10	0.8643	0.1357	1.50	0.9332	0.0668	1.90	0.9713	0.0287
0.31	0.6217	0.3783	0.71	0.7611	0.2389	1.11	0.8665	0.1335	1.51	0.9345	0.0655	1.91	0.9719	0.0281
0.32	0.6255	0.3745	0.72	0.7642	0.2358	1.12	0.8686	0.1314	1.52	0.9357	0.0643	1.92	0.9726	0.0274
0.33	0.6293	0.3707	0.73	0.7673	0.2327	1.13	0.8708	0.1292	1.53	0.9370	0.0630	1.93	0.9732	0.0268
0.34	0.6331	0.3669	0.74	0.7704	0.2296	1.14	0.8729	0.1271	1.54	0.9382	0.0618	1.94	0.9738	0.0262
0.35	0.6368	0.3632	0.75	0.7734	0.2266	1.15	0.8749	0.1251	1.55	0.9394	0.0606	1.95	0.9744	0.0256
0.36	0.6406	0.3594	0.76	0.7764	0.2236	1.16	0.8770	0.1230	1.56	0.9406	0.0594	**1.96**	**0.9750**	**0.0250**
0.37	0.6443	0.3557	0.77	0.7794	0.2206	1.17	0.8790	0.1210	1.57	0.9418	0.0582	1.97	0.9756	0.0244
0.38	0.6480	0.3520	0.78	0.7823	0.2177	1.18	0.8810	0.1190	1.58	0.9429	0.0571	1.98	0.9761	0.0239
0.39	0.6517	0.3483	0.79	0.7852	0.2148	1.19	0.8830	0.1170	1.59	0.9441	0.0559	1.99	0.9767	0.0233

Figure 34: Statistical table for N(0, 1).

Critical Values of the t Distribution

df	2-tailed testing			1-tailed testing		
	• •			• •		
	0.1	0.05	0.01	0.1	0.05	0.01
5	2.015	2.571	4.032	1.476	2.015	3.365
6	1.943	2.447	3.707	1.440	1.943	3.143
7	1.895	2.365	3.499	1.415	1.895	2.998
8	1.860	2.306	3.355	1.397	1.860	2.896
9	1.833	2.262	3.250	1.383	1.833	2.821
10	1.812	2.228	3.169	1.372	1.812	2.764
11	1.796	2.201	3.106	1.363	1.796	2.718
12	1.782	2.179	3.055	1.356	1.782	2.681
13	1.771	2.160	3.012	1.350	1.771	2.650
14	1.761	2.145	2.977	1.345	1.761	2.624
15	1.753	2.131	2.947	1.341	1.753	2.602
16	1.746	2.120	2.921	1.337	1.746	2.583
17	1.740	2.110	2.898	1.333	1.740	2.567
18	1.734	2.101	2.878	1.330	1.734	2.552
19	1.729	2.093	2.861	1.328	1.729	2.539
20	1.725	2.086	2.845	1.325	1.725	2.528
21	1.721	2.080	2.831	1.323	1.721	2.518
22	1.717	2.074	2.819	1.321	1.717	2.508
23	1.714	2.069	2.807	1.319	1.714	2.500
24	1.711	2.064	2.797	1.318	1.711	2.492
25	1.708	2.060	2.787	1.316	1.708	2.485
26	1.706	2.056	2.779	1.315	1.706	2.479
27	1.703	2.052	2.771	1.314	1.703	2.473
28	1.701	2.048	2.763	1.313	1.701	2.467
29	1.699	2.045	2.756	1.311	1.699	2.462
30	1.697	2.042	2.750	1.310	1.697	2.457
40	1.684	2.021	2.704	1.303	1.684	2.423
50	1.676	2.009	2.678	1.299	1.676	2.403
60	1.671	2.000	2.660	1.296	1.671	2.390
80	1.664	1.990	2.639	1.292	1.664	2.374
100	1.660	1.984	2.626	1.290	1.660	2.364
120	1.658	1.980	2.617	1.289	1.658	2.358
• •	1.645	1.960	2.576	1.282	1.645	2.327

Figure 35: Statistical table for Student's $t(r)$.

Critical Values of the F Distribution
($\alpha = .05$)

df within	df between										
	1	2	3	4	5	6	7	8	12	24	∞
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.68	4.53	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.79	2.61	2.41
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.69	2.51	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.20	2.01	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.18	1.98	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.16	1.96	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.12	1.91	1.66
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	1.92	1.70	1.39
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	1.88	1.65	1.33
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.85	1.63	1.28
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.83	1.61	1.26
∞	3.84	3.00	2.61	2.37	2.22	2.10	2.01	1.94	1.75	1.52	1.00

Figure 36: Statistical table for $F(m, n)$ at the 5% level.

Critical Values of the F Distribution
($\alpha = .01$)

df within	df between										
	1	2	3	4	5	6	7	8	12	24	∞
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	9.89	9.47	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.71	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	3.96	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	3.80	3.43	3.01
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.46	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.17	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	2.90	2.52	2.07
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	2.87	2.49	2.04
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.66	2.29	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.50	2.12	1.60
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.42	2.03	1.50
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.37	1.98	1.43
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.34	1.95	1.38
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.19	1.79	1.00

Figure 37: Statistical table for $F(m, n)$ at the 1% level.

Critical Values of the χ^2 Distribution

df	Area in the Upper Tail					
	0.99	0.95	0.9	0.1	**0.05**	**0.01**
1	0.000	0.004	0.016	2.706	3.841	6.635
2	0.020	0.103	0.211	4.605	5.991	9.210
3	0.115	0.352	0.584	6.251	7.815	11.345
4	0.297	0.711	1.064	7.779	9.488	13.277
5	0.554	1.145	1.610	9.236	11.070	15.086
6	0.872	1.635	2.204	10.645	12.592	16.812
7	1.239	2.167	2.833	12.017	14.067	18.475
8	1.646	2.733	3.490	13.362	15.507	20.090
9	2.088	3.325	4.168	14.684	16.919	21.666
10	2.558	3.940	4.865	15.987	18.307	23.209
11	3.053	4.575	5.578	17.275	19.675	24.725
12	3.571	5.226	6.304	18.549	21.026	26.217
13	4.107	5.892	7.042	19.812	22.362	27.688
14	4.660	6.571	7.790	21.064	23.685	29.141
15	5.229	7.261	8.547	22.307	24.996	30.578
16	5.812	7.962	9.312	23.542	26.296	32.000
17	6.408	8.672	10.085	24.769	27.587	33.409
18	7.015	9.390	10.865	25.989	28.869	34.805
19	7.633	10.117	11.651	27.204	30.144	36.191
20	8.260	10.851	12.443	28.412	31.410	37.566
21	8.897	11.591	13.240	29.615	32.671	38.932
22	9.542	12.338	14.041	30.813	33.924	40.289
23	10.196	13.091	14.848	32.007	35.172	41.638
24	10.856	13.848	15.659	33.196	36.415	42.980
25	11.524	14.611	16.473	34.382	37.652	44.314

Figure 38: Statistical table for $\chi^2(q)$.