Discrete choice modelling

- This problem set deals with maximum-likelihood estimation and asymptotic inference for binary choice models applied to cross-sectional data.
- We use the grade data from Spector and Mazzeo (1980, Journal of Economic Education), that is available as grades.txt on the website.
- There are n = 32 observations (obs) on individuals' grade point average (gpa), score on a pretest that indicates initial knowledge of the subject matter (tuce), an indicator of exposure to a new teaching method (psi), and an indicator of whether examination grades improved (1) or worsened (0) (grade).
- An important question from a policy-making standpoint is: do grades improve (grade = 1) after exposure to the new teaching method (psi = 1)?
- Refer to figures 1–33, and attempt the following:
 - Import the data, and perform a careful preliminary analysis of the variables. Briefly examine scatter plots, and simple descriptive statistics, to gain intuition about the behaviour of the data. What features of interest do you see?

What is the range of the grade point average and the pretest score? What percentage of the individuals had an improved grade? What percentage of the individuals had *not* been exposed to the new teaching method before the exam?

2. Estimate a probit model (probability of grade improvement as a function of explanatory variables: constant, grade point average, score on pretest, and exposure to new teaching method), and label the equation "probit_eqn":

$$Prob(grade = 1) = \Phi(\beta_0 + \beta_1 \text{ gpa} + \beta_2 \text{ tuce} + \beta_3 \text{ psi})$$

What are the *signs* of the estimated coefficients? Let y refer to the dependent variable throughout, and x to the vector of explanatory variables (constant, gpa, tuce, and psi). The marginal effects are:

$$\frac{\partial \operatorname{Prob}(y=1)}{\partial x} = \frac{\partial \operatorname{F}(x'\beta)}{\partial x} = f(x'\beta) \ \beta,$$

where for the probit model, $F(\cdot) = \Phi(\cdot)$, the normal N(0,1) cumulative distribution function, and $f(\cdot) = \phi(\cdot)$ is the density function. For obs = 1 and obs = 10, compute the marginal effects manually.

What do you notice about $\partial \widehat{\text{Prob}}(y=1)/\partial x$ (compare the results for obs = 1 and obs = 10 to answer this question)? [This illustrates one of the difficulties of interpreting marginal effects with binary choice models.]

Compute the sample means of gpa, tuce and psi. Then, manually calculate $\partial \widehat{\text{Prob}}(y=1)/\partial x$ given that $x=\bar{x}$ (the sample mean), where the sample mean of the constant is simply 1. Interpret your result.

3. Estimate a probit model of grade on constant, gpa, tuce and psi, using the Huber-White robust asymptotic covariance matrix $\widehat{\text{AVar}}(\widehat{\beta}) = \widehat{H}^{-1}\widehat{B}\widehat{H}^{-1} =$

$$\left[\frac{\partial^2 \ln L}{\partial \beta \partial \beta'} \middle| \beta = \widehat{\beta}\right]^{-1} \left[\sum_{i=1}^n \left(\frac{\phi(x'_i \beta)(y_i - \Phi(x'_i \beta))}{\Phi(x'_i \beta)(1 - \Phi(x'_i \beta))} \right)^2 x_i x'_i \middle| \beta = \widehat{\beta} \right] \left[\frac{\partial^2 \ln L}{\partial \beta \partial \beta'} \middle| \beta = \widehat{\beta} \right]^{-1}$$

and name this equation "probit_eqn_robust". What are the numerical values of the estimated standard errors $\widehat{se}(\widehat{\beta}) := (\widehat{AVar}(\widehat{\beta}))^{1/2}$?

4. The asymptotic covariance matrix of the estimated *probabilities* $\widehat{\text{Prob}}(y=1) = F(x'\widehat{\beta})$ are given by the "delta method" as (with an abuse of notation):

$$\widehat{\operatorname{AVar}}(F(x'\widehat{\beta})) = \left(\frac{\partial F(x'\widehat{\beta})}{\partial \widehat{\beta}'}\right) \widehat{\operatorname{AVar}}(\widehat{\beta}) \left(\frac{\partial F(x'\widehat{\beta})}{\partial \widehat{\beta}'}\right)',$$

where

$$\frac{\partial F(x'\widehat{\beta})}{\partial \widehat{\beta}} = \left(\frac{\partial F(x'\widehat{\beta})}{\partial (x'\widehat{\beta})}\right) \left(\frac{\partial (x'\widehat{\beta})}{\partial \widehat{\beta}}\right) = f(x'\widehat{\beta}) \ x,$$

so that

$$\widehat{\operatorname{AVar}}(F(x'\widehat{\beta})) = (f(x'\widehat{\beta}))^2 x' \widehat{\operatorname{AVar}}(\widehat{\beta}) x_{z'}$$

which depends upon $f(\cdot)$ and x.

Set $x = \bar{x}$ (the sample mean of the explanatory variables). Then, using the matrix $\widehat{\text{AVar}}(\widehat{\beta})$ from part 3 (see figures 7 and 8)), with $f(\cdot)$ and $F(\cdot)$ the density and distribution functions of the normal N(0,1), compute the scalar $\widehat{\text{AVar}}(F(\bar{x}'\widehat{\beta}))$ [Hint: this is equal to 0.0791365 times $(f(\bar{x}'\widehat{\beta}))^2$].

- 5. Using the robust standard errors from $\widehat{\text{AVar}}(\widehat{\beta})$, test the null hypothesis H_0 : $\beta_2 = 0$ (the coefficient on tuce), against the two-sided alternative, at the 95% level of significance. Use both a t test (compute this yourself, and justify your choice of critical value), and a Wald test (compute this using the software).
- 6. Plot the estimated residuals and the fitted probabilities. Does this give you any intuition regarding the quality of the probit model fit to the data? How would you test whether the probability Prob(y = 1) is equal to a constant, against a one-sided alternative? (You do not need to perform the test).
- 7. Compute the likelihood ratio statistic:

$$LR = -2(\ln \widehat{L}_0 - \ln \widehat{L}) \sim \chi^2(q),$$

where $\ln \hat{L}$ is the maximized log-likelihood, and $\ln \hat{L}_0$ is the maximized loglikelihood under the restriction that all coefficients except the constant are zero, and q is the number of restrictions imposed under the null that the restricted model is true. Test the null hypothesis that the restricted model is true, against the alternative that it is not, at the 95% level of significance.

8. Show that:

$$\ln \widehat{L}_0 = n[\bar{y}\ln\bar{y} + (1-\bar{y})\ln(1-\bar{y})]_{,}$$

where \bar{y} is the proportion of all observations that have dependent variable equal to 1 (we saw in part 1 that $\bar{y} \approx 0.343750$, i.e., the proportion of students with a grade improvement (grade = 1)).

Compute $\ln \hat{L}_0$, and compare the result with that in figure 10.

9. McFadden's pseudo- R^2 is defined as:

$$R_M^2 = 1 - \frac{\ln \widehat{L}}{\ln \widehat{L}_0}.$$

Compute this, and compare with the result in figure 10. Why is $R_M^2 < 1$?

10. Estimate a logit model (probability of grade improvement as a function of explanatory variables: constant, grade point average, score on pretest, and exposure to new teaching method), and label the equation "logit eqn":

$$Prob(grade = 1) = \Lambda(\beta_0 + \beta_1 \text{ gpa} + \beta_2 \text{ tuce} + \beta_3 \text{ psi}).$$

What are the *signs* of the estimated coefficients? Note that:

$$\frac{\partial \operatorname{Prob}(y=1)}{\partial x} = \frac{\partial \operatorname{F}(x'\beta)}{\partial x} = f(x'\beta) \ \beta,$$

where for the logit model, $F(x'\beta) = \Lambda(x'\beta) = (e^{x'\beta})/(1 + e^{x'\beta})$, the standard logistic cumulative distribution function. We have seen that:

$$\frac{\partial \widehat{\mathrm{Prob}}(y=1)}{\partial x} = \Lambda(x'\widehat{\beta})[1 - \Lambda(x'\widehat{\beta})]\widehat{\beta}.$$

Calculate the marginal effects, given that $x = \bar{x}$ (the sample mean), where we note that the sample mean of the constant is simply 1.

11. For the logit model, the probability $p := \operatorname{Prob}(y = 1)$ is defined as:

$$p := \frac{e^{x'\beta}}{1 + e^{x'\beta}}.$$

Show that:

$$\frac{p}{1-p} = e^{x'\beta}$$

This quantity is known as the *odds-ratio*, and can be useful when interpreting coefficients: it measures the probability of y = 1 relative to the probability of y = 0. For instance, if p/(1-p) = 2, then the 'odds' of (chance of) obtaining y = 1 are twice those of y = 0.

For the logit model in figure 11, evaluate the estimated odds ratio $\hat{p}/(1-\hat{p})$ at $x = \bar{x}$ (the sample mean).

- 12. Plot the estimated residuals and the fitted probabilities. Does this give you any intuition regarding the quality of the logit model fit to the data? Consider figure 12 (compare the fitted and actual probabilities, in particular) with your answer to part 11 in mind, and explain your findings.
- 13. Another interpretation for the logit model slope coefficients is that, if the *j*th regressor increases by one unit, then $e^{x'\beta}$ increases to $e^{x'\beta}e^{\beta_j}$, and so $p/(1-p) = e^{x'\beta}$ has increased by a factor e^{β_j} . For the model in figure 11, interpret the slope coefficient $\hat{\beta}_1 \approx 2.83$ in this way. [The resultant relative chance of obtaining y = 1 may seem huge.] Check the range of actual values of gpa, and comment.

The original odds-ratio for the logit model was $\hat{p}/(1-\hat{p}) \approx 0.338368$. Use your answer to the last question (on $e^{\hat{\beta}_1}$) to compute the new odds-ratio. What does this imply about the probability of grade improvement (= p) after a unit increase in gpa? [Hint: show that p is now 0.84, i.e., a unit increase in gpa (which is considerable), holding everything else constant, more than doubles the probability that y = 1.]

- 14. Since the marginal effects will depend upon the explanatory variables in general, refer to figures 13–33, to plot a probability response curve Prob(y = 1) against gpa, for psi = 0 and psi = 1, for both the probit model (figure 4) and the logit model (figure 11). That is, what is the probability of a grade increase given participation (or not) in the new teaching method, conditional on gpa (roughly, "intelligence"), holding all other variables at their sample means? [Hint: What is the effect of psi = 1 on the probabilities? What is the marginal effect of psi? What do you notice about the probability of grade increase after exposure to the new method, conditional on gpa score?]
- 15. Check the rule-of-thumb (which holds approximately):

$$\widehat{\beta}_{\text{logit}} \approx \left(\frac{\pi}{\sqrt{3}}\right) \widehat{\beta}_{\text{probit}}$$

using the estimated coefficients from figures 4 and 11.

16. Manually construct a 2 × 2 table of percentage "hits and misses", as a summary of the predictive ability of "probit_eqn". Use the decision rule: if $\widehat{\Phi} > 0.5$, $\widehat{\text{grade}} = 1$; and if $\widehat{\Phi} \leq 0.5$, $\widehat{\text{grade}} = 0$. What do you notice?

GPA	TUCE	PSI	GRADE
2.66	20	0	0
2.89	22	0	0
3.28	24	0	0
2.92	12	0	0
4.00	21	0	1
2.86	17	0	0
2.76	17	0	0
2.87	21	0	0
3.03	25	0	0
3.92	29	0	1
2.63	20	0	0
3.32	23	0	0
3.57	23	0	0
3.26	25	0	1
3.53	26	0	0
2.74	19	0	0
2.75	25	0	0
2.83	19	0	0
3.12	23	1	0
3.16	25	1	1
2.06	22	1	0
3.62	28	1	1
2.89	14	1	0
3.51	26	1	0
3.54	24	1	1
2.83	27	1	1
3.39	17	1	1
2.67	24	1	0
3.65	21	1	1
4.00	23	1	1
3.10	21	1	0
2.39	19	1	1
	GPA 2.66 2.89 3.28 2.92 4.00 2.86 2.76 2.87 3.03 3.92 2.63 3.32 3.57 3.26 3.53 2.74 2.75 2.83 3.12 3.16 2.06 3.62 2.89 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.54 2.83 3.51 3.59 2.65 4.00 3.10 2.83 3.57 3.53 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 2.65 3.59 3.59 3.59 3.59 3.51 3.59 3.59 3.59 3.59 3.59 3.51 3.59 3	GPATUCE2.66202.89223.28242.92124.00212.86172.76172.87213.03253.92292.63203.32233.57233.57233.57233.57233.57253.53262.74192.75252.83193.12233.16252.06223.62282.89143.51263.54242.83273.39172.67243.65214.00233.10212.3919	GPATUCEPSI 2.66 20 0 2.89 22 0 3.28 24 0 2.92 12 0 4.00 21 0 2.92 12 0 4.00 21 0 2.86 17 0 2.76 17 0 3.03 25 0 3.92 29 0 2.63 20 0 3.57 23 0 3.57 23 0 3.53 26 0 2.75 25 0 2.75 25 0 2.83 19 0 3.12 23 1 3.16 25 1 2.89 14 1 3.54 24 1 3.39 17 1 2.67 24 1 3.65 21 1 4.00 23 1 3.10 21 1

Figure 1: Raw data.

🗖 Group: UNT	ITLED Workfi	le: UNTITLED	Untitled					
View Proc Object	Print Name Freeze Sample Sheet Stats Spec							
	GPA	GRADE	PSI	TUCE				
Mean	3.117188	0.343750	0.437500	21.93750	~			
Median	3.065000	0.000000	0.000000	22.50000				
Maximum	4.000000	1.000000	1.000000	29.00000				
Minimum	2.060000	0.000000	0.000000	12.00000				
Std. Dev.	0.466713	0.482559	0.504016	3.901509				
Skewness	0.122657	0.657952	0.251976	-0.525110				
Kurtosis	2.570068	1.432900	1.063492	3.048305				
Jarque-Bera	0.326695	5.583204	5.338708	1.473728				
Probability	0.849296	0.061323	0.069297	0.478612				
Sum	99.75000	11.00000	14.00000	702.0000				
Sum Sq. Dev.	6.752447	7.218750	7.875000	471.8750				
Observations	32	32	32	32				
					×			
	<				≥ ;			

Figure	2:	Descriptive	statistics.
--------	----	-------------	-------------

Equation Estimation
Specification Options
Equation specification Binary dependent variable followed by list of regressors.
grade o gpa tuce psi
Binary estimation method:
Estimation settings
Method: BINARY - Binary choice (logit, probit, extreme value)
Sample: 1 32
OK Annuler

Figure 3: Estimate a probit model of grade on constant, gpa, tuce and psi.

Equation: PROBIT_	EQN Workf	ile: UNTITLE	DWntitled		×
View Proc Object Print N	Jame Freeze E	stimate Foreca	st Stats Resid	s	
Dependent Variable: G Method: ML - Binary F Date: 08/25/07 Time: Sample: 1 32 Included observations: Convergence achieved Covariance matrix con	RADE Probit (Quadra : 18:34 32 I after 5 iterati nputed using	itic hill climbir ons second deriva	ng) tives		
Variable	Coefficient	Std. Error	z-Statistic	Prob.	
C GPA TUCE PSI	-7.452320 1.625810 0.051729 1.426332	2.542472 0.693882 0.083890 0.595038	-2.931131 2.343063 0.616626 2.397045	0.0034 0.0191 0.5375 0.0165	
Mean dependent var S.E. of regression Sum squared resid Log likelihood Restr. log likelihood LR statistic (3 df) Probability(LR stat)	0.343750 0.386128 4.174660 -12.81880 -20.59173 15.54585 0.001405	S.D. depend Akaike info Schwarz crit Hannan-Qui Avg. log like McFadden F	lent var criterion terion nn criter. Ilihood R-squared	0.482559 1.051175 1.234392 1.111906 -0.400588 0.377478	
Obs with Dep=0 Obs with Dep=1	21 11	Total obs		32	
					_

Figure 4: Estimated probit model of grade on constant, gpa, tuce and psi.



Figure 5: Estimate a probit model of grade on constant, gpa, tuce and psi, using Huber-White robust covariances (these give the asymptotic covariance matrix $\widehat{AVar}(\widehat{\beta})$).

Equation: PROBIT_	EQN_ROBUS	T Workfile:	UNTITLED	u 🔳 🗖 🔀
View Proc Object Print N	Jame Freeze E	stimate Forecas	st Stats Resid	s
Dependent Variable: G Method: ML - Binary F Date: 11/19/07 Time: Sample: 1 32 Included observations: Convergence achieved QML (Huber/White) st	RADE Probit (Quadra : 15:34 32 after 5 iterati andard errors	ntic hill climbir ons & covariance	ng)	
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C GPA TUCE PSI	-7.452320 1.625810 0.051729 1.426332	2.544271 0.651510 0.069133 0.532765	-2.929059 2.495448 0.748256 2.677224	0.0034 0.0126 0.4543 0.0074
Mean dependent var S.E. of regression Sum squared resid Log likelihood Restr. log likelihood LR statistic (3 df) Probability(LR stat)	0.343750 0.386128 4.174660 -12.81880 -20.59173 15.54585 0.001405	S.D. depend Akaike info Schwarz crit Hannan-Quin Avg. log like McFadden F	lent var criterion terion nn criter. lihood R-squared	0.482559 1.051175 1.234392 1.111906 -0.400588 0.377478
Obs with Dep=0 Obs with Dep=1	21 11	Total obs		32

Figure 6: Estimated probit model of grade on constant, gpa, tuce and psi, using Huber-White robust covariances (this gives the square roots of the diagonal elements of the asymptotic covariance matrix $\widehat{AVar}(\widehat{\beta})$) as standard errors).

Equation: PROBIT_	EQN_ROBUS	тw	orkfile:	UNTITLED	u 🗖 🗖	\mathbf{X}
View Proc Object Print N	lame Freeze E	stimat	e Forecas	st Stats Resid	ls	
Representations Estimation Output Actual,Fitted,Residual Gradients		+ +	climbir	ıg)		
Covariance Matrix						
Coefficient Tests Residual Tests		*	ariance			
Dependent Variable Free	quencies		Error	z-Statistic	Prob.	
Categorical Regressor S Expectation-Prediction T Goodness-of-Fit Test (H Label	tats 'able osmer-Lemeshov	~)	14271 51510 59133 32765	-2.929059 2.495448 0.748256 2.677224	0.0034 0.0126 0.4543 0.0074	
Mean dependent var S.E. of regression Sum squared resid Log likelihood Restr. log likelihood LR statistic (3 df) Probability(LR stat)	0.343750 0.386128 4.174660 -12.81880 -20.59173 15.54585 0.001405	S.D. Aka Sch Han Avg. McF	. depend ike info warz crit nan-Quit log like adden F	lent var criterion terion nn criter. lihood R-squared	0.482559 1.051175 1.234392 1.111906 -0.400588 0.377478	
Obs with Dep=0 Obs with Dep=1	21 11	Tot	al obs		32	

Figure 7: Finding the asymptotic covariance matrix $\widehat{AVar}(\widehat{\beta})$.

🔲 Equation: PROBIT_EQN_ROBUST 🛛 Workfile: UNTITLEDVU 🔳 🗖 🔀								
View Proc Objec	t Print Name P	reeze Estimate	Forecast Stats	Resids				
	Coef	ficient Covaria	nce Matrix					
	С	GPA	TUCE	PSI				
С	6.473314	-1.299248	-0.092721	-0.241572		~		
GPA	-1.299248	0.424466	-0.004351	0.019652				
TUCE	-0.092721	-0.004351	0.004779	0.002505				
PSI	-0.241572	0.019652	0.002505	0.283839				
						_		
						Ξ		
	7							
	N			1111	1	**		

Figure 8: The asymptotic covariance matrix $\widehat{AVar}(\widehat{\beta})$.



Figure 9: Fitted probabilities and estimated residuals of the probit model.

Equation: PROBIT	EQN_ROBUS	T Workfile:	UNTITLED	Untitled	
View Proc Object Print M	Name Freeze E	stimate Foreca	st Stats Resid	s	
Dependent Variable: 0 Method: ML - Binary F Date: 11/19/07 Time Sample: 1 32 Included observations: Convergence achieved QML (Huber/White) st	GRADE Probit (Quadra : 15:34 : 32 I after 5 iterati andard errors	itic hill climbir ons & covariance	ng)		
Variable	Coefficient	Std. Error	z-Statistic	Prob.	
C GPA TUCE PSI	-7.452320 1.625810 0.051729 1.426332	2.544271 0.651510 0.069133 0.532765	-2.929059 2.495448 0.748256 2.677224	0.0034 0.0126 0.4543 0.0074	
Mean dependent var S.E. of regression Sum squared resid Log likelihood Restr. log likelihood LR statistic (3 df) Probability(LR stat)	0.343750 0.386128 4.174660 -12.81880 -20.59173 15.54585 0.001405	S.D. depend Akaike info Schwarz cri Hannan-Qui Avg. log like McFadden F	dent var criterion terion nn criter. lihood R-squared	0.482559 1.051175 1.234392 1.111906 -0.400588 0.377478	
Obs with Dep=0 Obs with Dep=1	21 11	Total obs		32	

Figure 10: Fitted log-likelihood, $\ln \hat{L}$ (Log likelihood) and fitted log-likelihood under the restriction that all coefficients except constant are zero, $\ln \hat{L}_0$ (Restr. log likelihood).

🛄 Equation: LOG	IT_EQN	Workt	file: UN	FITLED W	Untitled		
View Proc Object Pr	int Name	Freeze	Estimate	Forecast	Stats Resid	ds	
Dependent Variabl Method: ML - Bina Date: 08/25/07 T Sample: 1 32 Included observati Convergence achie Covariance matrix	le: GRAE ary Logit ime: 18:3 ons: 32 eved afte compute)E (Quadr: 34 r 5 itera ed usin(atic hill c ations g second	:limbing) I derivati	ves		
Variable	Coe	efficient	Std.	Error	z-Statistic	Prob.	
C GPA TUCE PSI	-13 2.0 0.0 2.3	.02135 326113 395158 378688	4.93 1.26 0.14 1.06	1317 2940 1554 4563	-2.640541 2.237725 0.672235 2.234426	0.0083 0.0252 0.5014 0.0255	
Mean dependent v S.E. of regression Sum squared resid Log likelihood Restr. log likelihoo LR statistic (3 df) Probability(LR stat	rar 0.3 0.3 d 4.1 -12 od -20 15 t) 0.0	343750 384716 144171 88963 59173 40419 001502	S.D. Akaik Schw Hann Avg. McFa	depende e info ci arz crite an-Quini log likeli idden R-	ent var riterion erion n criter. hood •squared	0.482559 1.055602 1.238819 1.116333 -0.402801 0.374038	
Obs with Dep=0 Obs with Dep=1		21 11	Tota	lobs		32	

Figure 11: Estimated logit model of grade on constant, gpa, tuce and psi.



Figure 12: Fitted probabilities and estimated residuals of the logit model.



Figure 13: Descriptive statistics on gpa: note the minimum and maximum, in particular.

🔛 EViews

File Edit Object View Proc Quick Options Window Help series gpa_plot=2+(4-2)*@trend/(@obs(@trend)-1)



Figure 14: Create an evenly-spaced series on the interval [2, 4].

🛄 Eq	uation: PR	OBIT_	EQN Wor	kfile: Ul	ITITLED	Wntitled		וא
View 8	rog Object	Print N	lame Freeze	Estimate	Forecas	t Stats Resid	s	
Dep Met Dat Sar Incl	Estimate Forecast () Make Resid Make Regr Make Grad	Fitted Pi dual Ser ressor G lient Gro	robability/Inde ies roup up	ex)	climbin	g)		
Cor Cov	Make Mode Update Co	el efs fron	n Equation		l derivat	ives		
	Variable		Coefficient	Std.	Error	z-Statistic	Prob.	
	C GPA TUCE PSI		-7.452320 1.625810 0.051729 1.426332	2.54 0.69 0.08 0.59	2472 3882 3890 5038	-2.931131 2.343063 0.616626 2.397045	0.0034 0.0191 0.5375 0.0165	
Mear S.E. Sum Log li Restr LR st Prob:	n dependen of regressi squared re ikelihood r. log likelih tatistic (3 d ability(LR s	t var on sid nood If) tat)	0.343750 0.386128 4.174660 -12.81880 -20.59173 15.54585 0.001405	S.D. Akaik Schw Hann Avg. McFa	dependo (e info c (arz crito an-Quin log likel adden R	ent var criterion erion in criter. ihood -squared	0.482559 1.051175 1.234392 1.111906 -0.400588 0.377478	
Obs v Obs v	with Dep=0 with Dep=1)	21 11	Tota	l obs		32	

Figure 15: From "probit_eqn", choose "Make Model".

Model: UNTITLED	Workfile: UNTITLED/Untitled	
View Proc Object Print M	Jame Freeze Solve Equations Variables Text	
Equations: 1		Baseline
PROBIT_EQN	Eq1: grade = F(gpa,psi,tuo	:e)
		2

Figure 16: Structure underlying "probit_eqn".

••• N	odel: UNTITLED Workfile:	UNTITLED/Untitled	
View	Print Name Freeze	Solve Equations Variables Text	
Equ	Solve Model		Baseline
∎F	Solve Control for Target	q1: grade = F(gpa,psi,tuce)	
	Links 🕨 🕨	Update All Links - Recompile model	
	Add Factors	Break All Links - Make all equations inline	
	Store series		
	Fetch series		
	Delete series		
	Make Group/Table		
	Make Graph		
20			5
<u>L</u>			1

Figure 17: Choose "Break All Links".

Model: UNTITLED Workfile: UNTITLEDWIntitled	
View Prog Object Print Name Freeze Solve Equations Variables Text	
Equations: 1	Baseline
📼 "grade = 1 - @cnorm(- " Eq1: grade = F(gpa, psi, tuce)	
	>

Figure 18: Further structure underlying "probit_eqn".

Model: UNTITLED	Workfile: UNTITLED/Untitled	
View Proc Object Print	Name Freeze Solve Equations Variables Text	
Equations		Baseline
Source Text	prm(" Eq1: grade = F(gpa,psi,tuce)	
Block Structure		
Solve Options		
Scenarios		
Solution Messages Trace Output		
Label		
<		>

Figure 19: Choose "Source Text".

Model: UNTITLED Workfile: UNTITLED/Untitled
View Proc Object Print Name Freeze Solve Equations Variables Text
View Proc Object Print Name Freeze Solve Equations Variables Text GRADE = 1 - @CNORM(- (-7.45231963284956 + 1.62581003959976 * GPA + 0.0517289448399536 * TUCE + 1.42633234278868 * PSI))

Figure 20: The estimated equation "probit_eqn".

Model: UNTITLED Workfile: UNTITLED/Untitled
View Proc Object Print Name Freeze Solve Equations Variables Text
View Proc Object Print Name Freeze Solve Equations Variables Text GRADE_0 = 1 - @CNORM(- (-7.45231963284956 + 1.62581003959976 * GPA_plot + 0.0517289448399536 * @mean(tuce) + 1.42633234278868 * 0))

Figure 21: Modify the estimated equation: rename as "GRADE_0", replace tuce by its sample mean, replace GPA by GPA_plot, and set psi = 0 (no exposure to new method).



Figure 22: Save modifications to "GRADE_0".

Vit Basic Options Stochastic Options Tracked Variables Diagnostics Solver GI Simulation type Deterministic Stochastic Stochastic Dynamics Dynamic solution Static solution Static solution Fit (static - no eq interactions) Structural (ignore ARMA) Solution sample Workfile sample used if left blank. Add/Delete Scenarios Add/Delete Scenarios 		Model Solution	> X
OK Annuler	S 0 0	Basic Options Stochastic Options 1 Simulation type Deterministic Stochastic Dynamics Dynamic solution Static solution Fit (static - no eq interactions) Structural (ignore ARMA) Solution sample Workfile sample used if left blank. 	Fracked Variables Diagnostics Solver Solution scenarios & output Active: Baseline Actuals Baseline Scenario 1 Solve for Alternate along with Active Alternate: Baseline Edit Scenario Options Add/Delete Scenarios

Figure 23: Select "Actuals".

Model: UNTITLED Workfile: UNTITLED/Untitled
View Proc Object Print Name Freeze Solve Equations Variables Text
Model: Untitled
Date: 08/25/07 Time: 18:53
Sample: 1 32 Solve Ontione:
Dynamic-Deterministic Simulation
Solver: Gauss-Seidel
Max iterations = 5000, Convergence = 1e-008
Scenario: Actuals Solve begin 18:53:29 Solve complete 18:53:29

Figure 24: Solve new model "GRADE_0".

Figure 25: Modify the estimated equation: rename as 'GRADE_1', replace tuce by its sample mean, replace GPA by GPA_plot, and set psi = 1 (exposure to new method).

Model: UNTITLED Workfile: UNTITLED/Untitled
View Proc Object Print Name Freeze Solve Equations Variables Text
Model: Untitled Date: 08/25/07 Time: 18:54 Sample: 1 32 Solve Options: Dynamic-Deterministic Simulation Solver: Gauss-Seidel Max iterations = 5000, Convergence = 1e-008
Scenario: Actuals Solve begin 18:54:12 Solve complete 18:54:12

Figure 26: Save modifications to "GRADE_1", select "Actuals", and solve the new model "GRADE_1".



Figure 27: Select "probit_eqn" and "New Object".



Figure 28: Select "Group".





🗖 Group	: UNTITLED	Workfile: UN1	TTLEDWntitle	d	
View Proc	Object Print Na	ame Freeze Def	ault 🗾 🛨 Sort	Transpose Edil	t+/- Smpl+/- InsDe
obs	GPA_PLOT	GRADE_0	GRADE_1		
1	2.000000	0.001085	0.050548		~
2	2.064516	0.001533	0.062432		
3	2.129032	0.002144	0.076390		
4	2.193548	0.002969	0.092604		
5	2.258065	0.004069	0.111233		
6	2.322581	0.005520	0.132402		
7	2.387097	0.007414	0.156196		
8	2.451613	0.009859	0.182646		
9	2.516129	0.012981	0.211729		
10	2.580645	0.016924	0.243357		
11	2.645161	0.021849	0.277376		
12	2.709677	0.027932	0.313568		
13	2.774194	0.035365	0.351649		
14	2.838710	0.044348	0.391281		
15	2.903226	0.055086	0.432077		
16	2.967742	0.067780	0.473610		
17	3.032258	0.082623	0.515432		
18	2 006774	n naa7an	0 667086		×
10	<			1111	≥:

r igure 50: Data in new grou	Figure	30:	Data	in	new	group
------------------------------	--------	-----	------	----	-----	-------

Group: UNTITLED W	orkfile: UN	TTLED\Untitle	ed	
🖉 👾 Proc Object Print Nam	e Freeze Def	ault 🛛 🔽 Sort	Transpose Edit+	-/- Smpl+/- InsDe
Group Members	GRADE_0	GRADE_1		
Spreadsheet	0.001085	0.050548		<u>^</u>
Dated Data Table	0.001533	0.062432		
Graph 🕨	Line			
Multiple Graphs 🔹 🕨	Area			
Descriptive Stats	Bar			
Tests of Equality	Spike			
N-Way Tabulation	Scatter	•		
Correlations	XY line	•	One X against a	ill Y's
Covariances	Error Bar		XY Pairs	
Principal Components	High-Low (Open-Close)		
	Pie			
Correlogram (1)	0.027932	0.313568		
Cross Correlation (2)	0.035365	0.351649		
Cointegration Test	0.044348	0.391281		
Unit Root Test	0.055086	0.432077		
Granger Causality	0.067780	0.473610		
Labol	0.082623	0.515432		
	n naa7an	0 667086	1	
10			1111	

Figure 31: Plot graph based on new group.



Figure 32: Probability response curves $\widehat{\text{Prob}}(y=1)$ against gpa, for psi = 0 and psi = 1, for the probit model.



Figure 33: Probability response curves $\widehat{\text{Prob}}(y=1)$ against gpa, for psi = 0 and psi = 1, for the logit model.

Ζ	Cum p	Tail p	Ζ	Cum p	Tail p	Ζ	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p
0,00	0,5000	0,5000	0.40	0.6554	0.3446	0.80	0.7881	0.2119	1.20	0,8849	0.1151	1.60	0.9452	0.0548
0.01	0.5040	0,4960	0.41	0.6591	0,3409	0,81	0.7910	0.2090	1.21	0,8869	0.1131	1,61	0.9463	0.0537
0.02	0,5080	0.4920	0.42	0.6628	0,3372	0.82	0.7939	0,2061	1.22	0,8888	0.1112	1,62	0.9474	0.0526
0.03	0.5120	0.4880	0.43	0.6664	0,3336	0.83	0.7967	0,2033	1.23	0,8907	0,1093	1,63	0.9484	0.0516
0.04	0,5160	0.4840	0.44	0.6700	0,3300	0.84	0.7995	0,2005	1.24	0,8925	0.1075	1,64	0.9495	0.0505
0.05	0.5199	0.4801	0.45	0.6736	0.3264	0.85	0.8023	0.1977	1.25	0.8944	0.1056	1.65	0,9505	0.0495
0.06	0.5239	0.4761	0.46	0,6772	0,3228	0,86	0.8051	0.1949	1.26	0,8962	0,1038	1,66	0.9515	0.0485
0.07	0.5279	0.4721	0.47	0,6808	0.3192	0.87	0.8078	0.1922	1.27	0,8980	0,1020	1,67	0.9525	0.0475
0.08	0,5319	0.4681	0.48	0.6844	0.3156	0.88	0.8106	0.1894	1.28	0,8997	0,1003	1,68	0,9535	0.0465
0.09	0,5359	0.4641	0.49	0.6879	0.3121	0.89	0.8133	0,1867	1.29	0.9015	0.0985	1,69	0.9545	0.0455
0.10	0.5398	0.4602	0,50	0.6915	0.3085	0,90	0.8159	0.1841	1.30	0,9032	0,0968	1,70	0.9554	0.0446
0.11	0,5438	0.4562	0.51	0.6950	0,3050	0.91	0.8186	0.1814	1.31	0,9049	0.0951	1,71	0.9564	0.0436
0.12	0,5478	0.4522	0.52	0.6985	0.3015	0.92	0.8212	0.1788	1.32	0,9066	0.0934	1,72	0.9573	0.0427
0.13	0,5517	0.4483	0.53	0.7019	0.2981	0,93	0.8238	0.1762	1.33	0,9082	0.0918	1,73	0.9582	0.0418
0.14	0,5557	0.4443	0.54	0,7054	0.2946	0,94	0.8264	0.1736	1.34	0,9099	0.0901	1.74	0.9591	0.0409
0.15	0,5596	0.4404	0,55	0,7088	0.2912	0,95	0.8289	0.1711	1.35	0.9115	0.0885	1,75	0.9599	0.0401
0.16	0,5636	0.4364	0,56	0.7123	0.2877	0,96	0.8315	0.1685	1.36	0.9131	0,0869	1,76	0,9608	0.0392
0.17	0.5675	0.4325	0,57	0.7157	0.2843	0.97	0.8340	0.1660	1.37	0.9147	0.0853	1,77	0,9616	0.0384
0.18	0.5714	0.4286	0.58	0.7190	0.2810	0,98	0.8365	0.1635	1.38	0.9162	0,0838	1.78	0.9625	0.0375
0.19	0.5753	0.4247	0.59	0.7224	0.2776	0,99	0.8389	0.1611	1.39	0.9177	0.0823	1.79	0.9633	0.0367
0,20	0,5793	0.4207	0,60	0.7257	0.2743	1.00	0.8413	0.1587	1.40	0.9192	0,0808	1,80	0.9641	0.0359
0.21	0,5832	0.4168	0.61	0.7291	0.2709	1.01	0.8438	0.1562	1.41	0.9207	0.0793	1.81	0.9649	0.0351
0.22	0.5871	0.4129	0.62	0.7324	0.2676	1.02	0.8461	0.1539	1.42	0.9222	0,0778	1.82	0,9656	0.0344
0.23	0.5910	0.4090	0.63	0,7357	0.2643	1.03	0.8485	0.1515	1.43	0.9236	0.0764	1.83	0.9664	0.0336
0.24	0.5948	0.4052	0.64	0,7389	0.2611	1.04	0.8508	0.1492	1.44	0.9251	0.0749	1.84	0.9671	0.0329
0.25	0.5987	0.4013	0.65	0.7422	0,2578	1.05	0.8531	0.1469	1.45	0.9265	0.0735	1,85	0.9678	0.0322
0,26	0.6026	0,3974	0,66	0.7454	0,2546	1,06	0.8554	0.1446	1.46	0,9279	0.0721	1,86	0,9686	0.0314
0.27	0.6064	0,3936	0,67	0,7486	0.2514	1.07	0.8577	0.1423	1.47	0,9292	0,0708	1,87	0,9693	0.0307
0,28	0.6103	0,3897	0,68	0,7517	0,2483	1.08	0.8599	0.1401	1.48	0,9306	0.0694	1,88	0,9699	0.0301
0,29	0.6141	0.3859	0.69	0,7549	0,2451	1.09	0.8621	0.1379	1.49	0,9319	0.0681	1.89	0,9706	0.0294
0,30	0.6179	0.3821	0.70	0,7580	0,2420	1.10	0.8643	0.1357	1.50	0,9332	0,0668	1,90	0.9713	0.0287
0.31	0.6217	0,3783	0.71	0,7611	0,2389	1.11	0.8665	0.1335	1.51	0.9345	0.0655	1,91	0,9719	0.0281
0,32	0.6255	0.3745	0.72	0.7642	0,2358	1.12	0,8686	0.1314	1.52	0,9357	0.0643	1,92	0.9726	0.0274
0,33	0.6293	0.3707	0.73	0,7673	0,2327	1.13	0.8708	0.1292	1.53	0,9370	0,0630	1,93	0.9732	0.0268
0.34	0,6331	0.3669	0.74	0.7704	0,2296	1.14	0.8729	0.1271	1.54	0,9382	0,0618	1,94	0,9738	0.0262
0,35	0,6368	0.3632	0.75	0.7734	0,2266	1.15	0.8749	0.1251	1.55	0.9394	0,0606	1.95	0.9744	0.0256
0,36	0,6406	0.3594	0.76	0.7764	0,2236	1.16	0,8770	0.1230	1.56	0.9406	0.0594	1.96	0,9750	0.0250
0.37	0.6443	0.3557	0.77	0.7794	0,2206	1.17	0.8790	0.1210	1.57	0.9418	0.0582	1.97	0.9756	0.0244
0,38	0,6480	0,3520	0.78	0.7823	0.2177	1.18	0,8810	0.1190	1.58	0.9429	0.0571	1.98	0,9761	0.0239
0,39	0.6517	0.3483	0,79	0,7852	0.2148	1.19	0.8830	0.1170	1.59	0.9441	0,0559	1,99	0,9767	0.0233

Areas Under the Normal Curve

Figure 34: Statistical table for N(0, 1).

	2-	tailed testir	1g	1-tailed testing					
df									
	0.1	0.05	0.01	0.1	0.05	0.01			
5	2.015	2.571	4.032	1.476	2.015	3.365			
6	1.943	2.447	3.707	1.440	1.943	3.143			
7	1.895	2.365	3.499	1.415	1.895	2.998			
8	1.860	2.306	3.355	1.397	1.860	2.896			
9	1.833	2.262	3.250	1.383	1.833	2.821			
10	1.812	2.228	3.169	1.372	1.812	2.764			
11	1.796	2.201	3.106	1.363	1.796	2.718			
12	1.782	2.179	3.055	1.356	1.782	2.681			
13	1.771	2.160	3.012	1.350	1.771	2.650			
14	1.761	2.145	2.977	1.345	1.761	2.624			
15	1.753	2.131	2.947	1.341	1.753	2.602			
16	1.746	2.120	2.921	1.337	1.746	2.583			
17	1.740	2.110	2.898	1.333	1.740	2.567			
18	1.734	2.101	2.878	1.330	1.734	2.552			
19	1.729	2.093	2.861	1.328	1.729	2.539			
20	1.725	2.086	2.845	1.325	1.725	2.528			
21	1.721	2.080	2.831	1.323	1.721	2.518			
22	1.717	2.074	2.819	1.321	1.717	2.508			
23	1.714	2.069	2.807	1.319	1.714	2.500			
24	1.711	2.064	2.797	1.318	1.711	2.492			
25	1.708	2.060	2.787	1.316	1.708	2.485			
26	1.706	2.056	2.779	1.315	1.706	2.479			
27	1.703	2.052	2.771	1.314	1.703	2.473			
28	1.701	2.048	2.763	1.313	1.701	2.467			
29	1.699	2.045	2.756	1.311	1.699	2.462			
30	1.697	2.042	2.750	1.310	1.697	2.457			
40	1.684	2.021	2.704	1.303	1.684	2.423			
50	1.676	2.009	2.678	1.299	1.676	2.403			
60	1.671	2.000	2.660	1.296	1.671	2.390			
80	1.664	1.990	2.639	1.292	1.664	2.374			
100	1.660	1.984	2.626	1.290	1.660	2.364			
120	1.658	1.980	2.617	1.289	1.658	2.358			
	1.645	1.960	2,576	1.282	1.645	2.327			

Critical Values of the <u>t</u> Distribution

Figure 35: Statistical table for Student's t(r).

Critical Values of the <u>F</u> Distribution ($\alpha = .05$)

df	df between										
within	1	2	3	4	5	6	7	8	12	24	8
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.68	4.53	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.79	2.61	2.41
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.69	2.51	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.20	2.01	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.18	1.98	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.16	1.96	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.12	1.91	1.66
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	1.92	1.70	1.39
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	1.88	1.65	1.33
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.85	1.63	1.28
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.83	1.61	1.26
8	3.84	3.00	2.61	2.37	2.22	2.10	2.01	1.94	1.75	1.52	1.00

Figure 36: Statistical table for F(m, n) at the 5% level.

Critical Values of the <u>F</u> Distribution ($\alpha = .01$)

df	df between										
within	1	2	3	4	5	6	7	8	12	24	8
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	9.89	9.47	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.71	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	3.96	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	3.80	3.43	3.01
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.46	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.17	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	2.90	2.52	2.07
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	2.87	2.49	2.04
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.66	2.29	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.50	2.12	1.60
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.42	2.03	1.50
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.37	1.98	1.43
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.34	1.95	1.38
00	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.19	1.79	1.00

Figure 37: Statistical table for F(m,n) at the 1% level.

36	Area in the Upper Tail											
ai .	0.99	0.95	0.9	0.1	0.05	0.01						
1	0.000	0.004	0.016	2.706	3.841	6.635						
2	0.020	0.103	0.211	4.605	5.991	9.210						
3	0.115	0.352	0.584	6.251	7.815	11.345						
4	0.297	0.711	1.064	7.779	9.488	13.277						
5	0.554	1.145	1.610	9.236	11.070	15.086						
6	0.872	1.635	2.204	10.645	12.592	16.812						
7	1.239	2.167	2.833	12.017	14.067	18.475						
8	1.646	2.733	3.490	13.362	15.507	20.090						
9	2.088	3.325	4.168	14.684	16.919	21.666						
10	2.558	3.940	4.865	15.987	18.307	23.209						
11	3.053	4.575	5.578	17.275	19.675	24.725						
12	3.571	5.226	6.304	18.549	21.026	26.217						
13	4.107	5.892	7.042	19.812	22.362	27.688						
14	4.660	6.571	7.790	21.064	23.685	29.141						
15	5.229	7.261	8.547	22.307	24.996	30.578						
16	5.812	7.962	9.312	23.542	26.296	32.000						
17	6.408	8.672	10.085	24.769	27.587	33.409						
18	7.015	9.390	10.865	25.989	28.869	34.805						
19	7.633	10.117	11.651	27.204	30.144	36.191						
20	8.260	10.851	12.443	28.412	31.410	37.566						
21	8.897	11.591	13.240	29.615	32.671	38.932						
22	9.542	12.338	14.041	30.813	33.924	40.289						
23	10.196	13.091	14.848	32.007	35.172	41.638						
24	10.856	13.848	15.659	33.196	36.415	42.980						
25	11.524	14.611	16.473	34.382	37.652	44.314						

Critical Values of the χ^2 Distribution

Figure 38: Statistical table for $\chi^2(q)$.