IENAC22 / Econometrics 2 / Applied Problem Set 3

Topic: Autocorrelation

- This problem set deals with autocorrelation, and in particular detection (visually and by several diagnostic tests), robust inference (using Newey-West standard errors), and nonlinear least squares for model estimation with autocorrelated errors.

- We use annual U.S. macroeconomic time-series data over the period 1963–1982 ($t = 1, 2, \ldots, 20$), available as `investment.txt` on the website. Import the data into a **dated** workfile as `year` (date), `gnp` (output), `inv` (investment), `p` (price level) and `int` (interest rate). In what follows, T refers to the effective sample size.

- Refer to figures 1 – 3, and perform the following:

  1. Perform a brief descriptive analysis of the dataset. Do you notice any features of interest? Then, run a linear regression of `inv`/`p` on a constant, `gnp`/`p`, and `int` $- \Delta$`p`, where $\Delta \mathtt{p} := \mathtt{p}_t - \mathtt{p}_{t-1}$, and the EViews command for first-differencing is `d(p,1)`. Plot the estimated residuals $\widehat{u}_t$ against time, and interpret the results. Check for the normality of $\widehat{u}_t$ at the 90% level.[1] Finally, test $H_0 : \beta_2 = -8$ (where β_2 is the weight on `int` $- \Delta$`p`) against the two-sided H_1, at the 95% level.

  2. Re-run the above regression, using Newey-West robust (to heteroscedasticity and autocorrelation of unknown form) standard errors. Calculate the lag length $L = \text{integer}(4(T/100)^{2/9})$ that is used in the computation of the standard errors.

[1] Hint (optional, advanced): how many observations have been used in the model estimation? Suggestion: use the *asymptotic* $\chi^2_{0.90}(2)$ critical value, then use Figure 3 and/or Table 1 in the 'Finite-sample quantiles of the Jarque-Bera test' handout to approximate the *actual* 90% level critical value to 1 d.p.; does this change the test result?

What do you notice from the regression output? Again, test $H_0 : \beta_2 = -8$ against the two-sided alternative, at the 95% level. Explain your findings.

3. One of the most widely-used checks for autocorrelation is the Durbin-Watson d statistic, which is based on the principle that if the true disturbances are autocorrelated, then this should be revealed through autocorrelation of $\widehat{u}_t$:

$$d = \frac{\sum_{t=2}^{T}(\widehat{u}_t - \widehat{u}_{t-1})^2}{\sum_{t=1}^{T}\widehat{u}_t^2}.$$

This is related to the sample autocorrelation $\widehat{\rho} := (\sum_{t=2}^{T}\widehat{u}_t\widehat{u}_{t-1})/(\sum_{t=1}^{T}\widehat{u}_t^2)$:

$$d = 2(1-\widehat{\rho}) - \left(\frac{\widehat{u}_1^2 + \widehat{u}_T^2}{\sum_{t=1}^{T}\widehat{u}_t^2}\right).$$

(Show this.) If the sample is large, then:

$$d \approx 2(1-\widehat{\rho}),$$

and so $d \approx 2$ as $\widehat{\rho} = 0$ and $d > 2$ as $\widehat{\rho} < 0$ and $d < 2$ as $\widehat{\rho} > 0$. Practically, the distribution of d under the null hypothesis $H_0 :$ no autocorrelation ($\rho = 0$), depends upon on the explanatory variables in the regression, and the critical region does not come from a 'standard' distribution. The d test is not valid if lagged dependent variables are used as explanatory variables. Further, the d test can only be used to test for no autocorrelation against $H_1 :$ first-order autocorrelation. What is the interpretation of the d statistic from part 1?

4. An *asymptotic approximation* to the distribution of d (as $T \to \infty$) is:

$$d \sim \mathrm{N}(2, 4/T).$$

Use this result to formally test $H_0 : \rho = 0$ at the 95% level.

5. A more sophisticated check for autocorrelation is the Breusch-Godfrey test, which assesses the null of no autocorrelation up to lag p, against the alternative that $u_t \sim \text{AR}(p)$ or $u_t \sim \text{MA}(p)$ (the same test procedure is used for both). It consists of regressing $\widehat{u}_t$ on a constant, the original (non-constant) explanatory variables *and* $\widehat{u}_{t-1}, \ldots, \widehat{u}_{t-p}$, computing the R^2 from this regression, and noting:

$$T\,R^2 \sim \chi^2(p),$$

where T here is the *effective* sample size. This is equivalent to regressing $\widehat{u}_t$ on the part of the lagged residuals that is *unexplained by the explanatory variables* (if some fit is found, this will be due to the correlation between the current and lagged residuals: autocorrelation!). Interpret the results of this test statistic (computed using EViews), with $p = 4$.[2]

6. Regress `inv/p` on a constant, `gnp/p`, `int` $-$ Δ`p`, *and* AR(1) errors, using non-linear least squares (to include AR(1) errors in a model, use `ar(1)`; to include AR(2) errors, use `ar(1)` and `ar(2)` together, etc.). Compare the results to part 1 above, and interpret the AR(1) coefficient. Does the Breusch-Godfrey test suggest that explicit modelling of error dependence has removed the problem of autocorrelated errors?! (Hint: try various lags p).

- Note that this problem set is intended to introduce methods of treating autocorrelation, and does not deal with some fundamental time-series issues (such as possible nonstationarity of the series) that can be very important when modelling such data.

[2] Note that performing this test manually will give a different value for the computed test statistic, since EViews sets all presample values of the residuals to zero rather than dropping those observations, e.g. $\widehat{u}_0 = 0$, $\widehat{u}_{-1} = 0$, etc. Furthermore, a different choice of lag p may also change the test outcome!

Equation: EQ01 Workfile: APPLIED_PROBLEM_SET_3_...

View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Dependent Variable: INV/P
Method: Least Squares
Date: 09/19/09 Time: 15:02
Sample (adjusted): 2 20
Included observations: 19 after adjustments

	Coefficient	Std. Error	t-Statistic	Prob.
C	-33.37635	31.06374	-1.074447	0.2986
GNP/P	0.201524	0.036222	5.563540	0.0000
INT-D(P,1)	-2.934159	2.609294	-1.124503	0.2774

R-squared	0.825756	Mean dependent var	192.4258
Adjusted R-squared	0.803975	S.D. dependent var	37.62754
S.E. of regression	16.65948	Akaike info criterion	8.607775
Sum squared resid	4440.612	Schwarz criterion	8.756897
Log likelihood	-78.77386	Hannan-Quinn criter.	8.633012
F-statistic	37.91253	Durbin-Watson stat	1.430277
Prob(F-statistic)	0.000001		

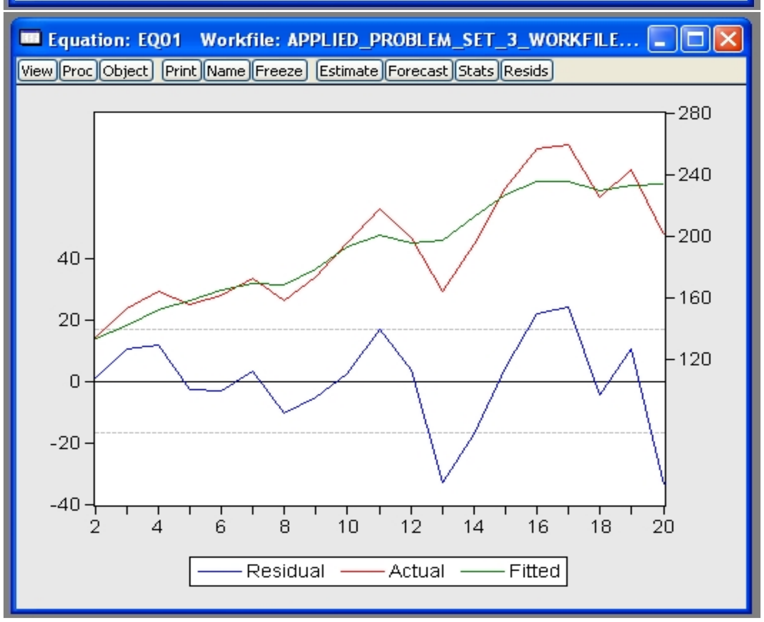

Figure 1: Ordinary least squares regression of $\texttt{inv}/\texttt{p}$ on a constant, $\texttt{gnp}/\texttt{p}$, and $\texttt{int} - \Delta\texttt{p}$, and actual and fitted residuals.

Equation: EQ02 Workfile: APPLIED_PROBLEM_SET_3_...

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: INV/P
Method: Least Squares
Date: 09/19/09 Time: 16:17
Sample (adjusted): 2 20
Included observations: 19 after adjustments
Newey-West HAC Standard Errors & Covariance (lag truncation=2)

	Coefficient	Std. Error	t-Statistic	Prob.
C	-33.37635	29.03188	-1.149645	0.2672
GNP/P	0.201524	0.033320	6.048164	0.0000
INT-D(P,1)	-2.934159	1.841019	-1.593769	0.1305

R-squared	0.825756	Mean dependent var	192.4258
Adjusted R-squared	0.803975	S.D. dependent var	37.62754
S.E. of regression	16.65948	Akaike info criterion	8.607775
Sum squared resid	4440.612	Schwarz criterion	8.756897
Log likelihood	-78.77386	Hannan-Quinn criter.	8.633012
F-statistic	37.91253	Durbin-Watson stat	1.430277
Prob(F-statistic)	0.000001		

Equation: EQ01 Workfile: APPLIED_PROBLEM_SET_3_...

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	5.321288	Prob. F(4,12)	0.0106
Obs*R-squared	12.15010	Prob. Chi-Square(4)	0.0163

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 09/19/09 Time: 23:19
Sample: 2 20
Included observations: 19
Presample missing value lagged residuals set to zero.

	Coefficient	Std. Error	t-Statistic	Prob.
C	78.26295	31.09556	2.516853	0.0271
GNP/P	-0.117293	0.042453	-2.762899	0.0172
INT-D(P,1)	10.21571	3.546754	2.880299	0.0138
RESID(-1)	-0.704523	0.343648	-2.050128	0.0629
RESID(-2)	-0.899918	0.391478	-2.298772	0.0403
RESID(-3)	-0.757747	0.336969	-2.248713	0.0441
RESID(-4)	-0.719722	0.373210	-1.928466	0.0778

R-squared	0.639479	Mean dependent var	-5.68E-14
Adjusted R-squared	0.459218	S.D. dependent var	15.70671
S.E. of regression	11.55038	Akaike info criterion	8.008623
Sum squared resid	1600.934	Schwarz criterion	8.356574
Log likelihood	-69.08192	Hannan-Quinn criter.	8.067510
F-statistic	3.547525	Durbin-Watson stat	1.483156
Prob(F-statistic)	0.029477		

Figure 2: Ordinary least squares regression of $\texttt{inv}/\texttt{p}$ on a constant, $\texttt{gnp}/\texttt{p}$, and $\texttt{int} - \Delta\texttt{p}$, with Newey-West standard errors, and results of Breusch-Godfrey test for autocorrelation.

Equation: EQ03 Workfile: APPLIED_PROBLEM_SET_3_...

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: INV/P
Method: Least Squares
Date: 09/20/09 Time: 00:05
Sample (adjusted): 3 20
Included observations: 18 after adjustments
Convergence achieved after 27 iterations

	Coefficient	Std. Error	t-Statistic	Prob.
C	-33.42004	49.39430	-0.676597	0.5097
GNP/P	0.198892	0.056279	3.534048	0.0033
INT-D(P,1)	-2.546250	3.964467	-0.642268	0.5311
AR(1)	0.227715	0.384625	0.592045	0.5633

R-squared	0.804130	Mean dependent var	195.6802
Adjusted R-squared	0.762157	S.D. dependent var	35.86148
S.E. of regression	17.48933	Akaike info criterion	8.754189
Sum squared resid	4282.272	Schwarz criterion	8.952049
Log likelihood	-74.78770	Hannan-Quinn criter.	8.781471
F-statistic	19.15860	Durbin-Watson stat	1.558166
Prob(F-statistic)	0.000032		

Equation: EQ03 Workfile: APPLIED_PROBLEM_SET_3_...

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	5.811335	Prob. F(4,10)	0.0111
Obs*R-squared	12.58571	Prob. Chi-Square(4)	0.0135

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 09/20/09 Time: 00:24
Sample: 3 20
Included observations: 18
Presample missing value lagged residuals set to zero.

	Coefficient	Std. Error	t-Statistic	Prob.
C	106.2078	49.28784	2.154849	0.0566
GNP/P	-0.146960	0.057510	-2.555360	0.0286
INT-D(P,1)	12.20695	4.006718	3.046620	0.0123
AR(1)	-0.544850	7.291098	-0.074728	0.9419
RESID(-1)	-0.439000	7.226590	-0.060748	0.9528
RESID(-2)	-0.938350	1.578883	-0.594312	0.5655
RESID(-3)	-0.871129	0.461685	-1.886846	0.0885
RESID(-4)	-0.877455	0.350189	-2.505662	0.0311

R-squared	0.699206	Mean dependent var	1.94E-05
Adjusted R-squared	0.488650	S.D. dependent var	15.87131
S.E. of regression	11.34937	Akaike info criterion	7.997304
Sum squared resid	1288.082	Schwarz criterion	8.393024
Log likelihood	-63.97573	Hannan-Quinn criter.	8.051868
F-statistic	3.320763	Durbin-Watson stat	1.589060
Prob(F-statistic)	0.042374		

Figure 3: Ordinary least squares regression of $\texttt{inv}/\texttt{p}$ on a constant, $\texttt{gnp}/\texttt{p}$, and $\texttt{int} - \Delta\texttt{p}$, with AR(1) errors, and results of the Breusch-Godfrey test for autocorrelation.

Areas Under the Normal Curve

Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p	Z	Cum p	Tail p
0.00	0.5000	0.5000	0.40	0.6554	0.3446	0.80	0.7881	0.2119	1.20	0.8849	0.1151	1.60	0.9452	0.0548
0.01	0.5040	0.4960	0.41	0.6591	0.3409	0.81	0.7910	0.2090	1.21	0.8869	0.1131	1.61	0.9463	0.0537
0.02	0.5080	0.4920	0.42	0.6628	0.3372	0.82	0.7939	0.2061	1.22	0.8888	0.1112	1.62	0.9474	0.0526
0.03	0.5120	0.4880	0.43	0.6664	0.3336	0.83	0.7967	0.2033	1.23	0.8907	0.1093	1.63	0.9484	0.0516
0.04	0.5160	0.4840	0.44	0.6700	0.3300	0.84	0.7995	0.2005	1.24	0.8925	0.1075	1.64	0.9495	0.0505
0.05	0.5199	0.4801	0.45	0.6736	0.3264	0.85	0.8023	0.1977	1.25	0.8944	0.1056	**1.65**	**0.9505**	**0.0495**
0.06	0.5239	0.4761	0.46	0.6772	0.3228	0.86	0.8051	0.1949	1.26	0.8962	0.1038	1.66	0.9515	0.0485
0.07	0.5279	0.4721	0.47	0.6808	0.3192	0.87	0.8078	0.1922	1.27	0.8980	0.1020	1.67	0.9525	0.0475
0.08	0.5319	0.4681	0.48	0.6844	0.3156	0.88	0.8106	0.1894	1.28	0.8997	0.1003	1.68	0.9535	0.0465
0.09	0.5359	0.4641	0.49	0.6879	0.3121	0.89	0.8133	0.1867	1.29	0.9015	0.0985	1.69	0.9545	0.0455
0.10	0.5398	0.4602	0.50	0.6915	0.3085	0.90	0.8159	0.1841	1.30	0.9032	0.0968	1.70	0.9554	0.0446
0.11	0.5438	0.4562	0.51	0.6950	0.3050	0.91	0.8186	0.1814	1.31	0.9049	0.0951	1.71	0.9564	0.0436
0.12	0.5478	0.4522	0.52	0.6985	0.3015	0.92	0.8212	0.1788	1.32	0.9066	0.0934	1.72	0.9573	0.0427
0.13	0.5517	0.4483	0.53	0.7019	0.2981	0.93	0.8238	0.1762	1.33	0.9082	0.0918	1.73	0.9582	0.0418
0.14	0.5557	0.4443	0.54	0.7054	0.2946	0.94	0.8264	0.1736	1.34	0.9099	0.0901	1.74	0.9591	0.0409
0.15	0.5596	0.4404	0.55	0.7088	0.2912	0.95	0.8289	0.1711	1.35	0.9115	0.0885	1.75	0.9599	0.0401
0.16	0.5636	0.4364	0.56	0.7123	0.2877	0.96	0.8315	0.1685	1.36	0.9131	0.0869	1.76	0.9608	0.0392
0.17	0.5675	0.4325	0.57	0.7157	0.2843	0.97	0.8340	0.1660	1.37	0.9147	0.0853	1.77	0.9616	0.0384
0.18	0.5714	0.4286	0.58	0.7190	0.2810	0.98	0.8365	0.1635	1.38	0.9162	0.0838	1.78	0.9625	0.0375
0.19	0.5753	0.4247	0.59	0.7224	0.2776	0.99	0.8389	0.1611	1.39	0.9177	0.0823	1.79	0.9633	0.0367
0.20	0.5793	0.4207	0.60	0.7257	0.2743	1.00	0.8413	0.1587	1.40	0.9192	0.0808	1.80	0.9641	0.0359
0.21	0.5832	0.4168	0.61	0.7291	0.2709	1.01	0.8438	0.1562	1.41	0.9207	0.0793	1.81	0.9649	0.0351
0.22	0.5871	0.4129	0.62	0.7324	0.2676	1.02	0.8461	0.1539	1.42	0.9222	0.0778	1.82	0.9656	0.0344
0.23	0.5910	0.4090	0.63	0.7357	0.2643	1.03	0.8485	0.1515	1.43	0.9236	0.0764	1.83	0.9664	0.0336
0.24	0.5948	0.4052	0.64	0.7389	0.2611	1.04	0.8508	0.1492	1.44	0.9251	0.0749	1.84	0.9671	0.0329
0.25	0.5987	0.4013	0.65	0.7422	0.2578	1.05	0.8531	0.1469	1.45	0.9265	0.0735	1.85	0.9678	0.0322
0.26	0.6026	0.3974	0.66	0.7454	0.2546	1.06	0.8554	0.1446	1.46	0.9279	0.0721	1.86	0.9686	0.0314
0.27	0.6064	0.3936	0.67	0.7486	0.2514	1.07	0.8577	0.1423	1.47	0.9292	0.0708	1.87	0.9693	0.0307
0.28	0.6103	0.3897	0.68	0.7517	0.2483	1.08	0.8599	0.1401	1.48	0.9306	0.0694	1.88	0.9699	0.0301
0.29	0.6141	0.3859	0.69	0.7549	0.2451	1.09	0.8621	0.1379	1.49	0.9319	0.0681	1.89	0.9706	0.0294
0.30	0.6179	0.3821	0.70	0.7580	0.2420	1.10	0.8643	0.1357	1.50	0.9332	0.0668	1.90	0.9713	0.0287
0.31	0.6217	0.3783	0.71	0.7611	0.2389	1.11	0.8665	0.1335	1.51	0.9345	0.0655	1.91	0.9719	0.0281
0.32	0.6255	0.3745	0.72	0.7642	0.2358	1.12	0.8686	0.1314	1.52	0.9357	0.0643	1.92	0.9726	0.0274
0.33	0.6293	0.3707	0.73	0.7673	0.2327	1.13	0.8708	0.1292	1.53	0.9370	0.0630	1.93	0.9732	0.0268
0.34	0.6331	0.3669	0.74	0.7704	0.2296	1.14	0.8729	0.1271	1.54	0.9382	0.0618	1.94	0.9738	0.0262
0.35	0.6368	0.3632	0.75	0.7734	0.2266	1.15	0.8749	0.1251	1.55	0.9394	0.0606	1.95	0.9744	0.0256
0.36	0.6406	0.3594	0.76	0.7764	0.2236	1.16	0.8770	0.1230	1.56	0.9406	0.0594	**1.96**	**0.9750**	**0.0250**
0.37	0.6443	0.3557	0.77	0.7794	0.2206	1.17	0.8790	0.1210	1.57	0.9418	0.0582	1.97	0.9756	0.0244
0.38	0.6480	0.3520	0.78	0.7823	0.2177	1.18	0.8810	0.1190	1.58	0.9429	0.0571	1.98	0.9761	0.0239
0.39	0.6517	0.3483	0.79	0.7852	0.2148	1.19	0.8830	0.1170	1.59	0.9441	0.0559	1.99	0.9767	0.0233

Figure 4: Statistical table for N(0, 1). These tables have been taken from: `http://fsweb.berry.edu/academic/education/vbissonnette/tables/tables.html`.

Critical Values of the t Distribution

df	2-tailed testing			1-tailed testing		
	• •			• •		
	0.1	0.05	0.01	0.1	0.05	0.01
5	2.015	2.571	4.032	1.476	2.015	3.365
6	1.943	2.447	3.707	1.440	1.943	3.143
7	1.895	2.365	3.499	1.415	1.895	2.998
8	1.860	2.306	3.355	1.397	1.860	2.896
9	1.833	2.262	3.250	1.383	1.833	2.821
10	1.812	2.228	3.169	1.372	1.812	2.764
11	1.796	2.201	3.106	1.363	1.796	2.718
12	1.782	2.179	3.055	1.356	1.782	2.681
13	1.771	2.160	3.012	1.350	1.771	2.650
14	1.761	2.145	2.977	1.345	1.761	2.624
15	1.753	2.131	2.947	1.341	1.753	2.602
16	1.746	2.120	2.921	1.337	1.746	2.583
17	1.740	2.110	2.898	1.333	1.740	2.567
18	1.734	2.101	2.878	1.330	1.734	2.552
19	1.729	2.093	2.861	1.328	1.729	2.539
20	1.725	2.086	2.845	1.325	1.725	2.528
21	1.721	2.080	2.831	1.323	1.721	2.518
22	1.717	2.074	2.819	1.321	1.717	2.508
23	1.714	2.069	2.807	1.319	1.714	2.500
24	1.711	2.064	2.797	1.318	1.711	2.492
25	1.708	2.060	2.787	1.316	1.708	2.485
26	1.706	2.056	2.779	1.315	1.706	2.479
27	1.703	2.052	2.771	1.314	1.703	2.473
28	1.701	2.048	2.763	1.313	1.701	2.467
29	1.699	2.045	2.756	1.311	1.699	2.462
30	1.697	2.042	2.750	1.310	1.697	2.457
40	1.684	2.021	2.704	1.303	1.684	2.423
50	1.676	2.009	2.678	1.299	1.676	2.403
60	1.671	2.000	2.660	1.296	1.671	2.390
80	1.664	1.990	2.639	1.292	1.664	2.374
100	1.660	1.984	2.626	1.290	1.660	2.364
120	1.658	1.980	2.617	1.289	1.658	2.358
• •	1.645	1.960	2.576	1.282	1.645	2.327

Figure 5: Statistical table for Student's $t(r)$.

Critical Values of the F Distribution
(α = .05)

df	df between										
within	1	2	3	4	5	6	7	8	12	24	∞
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.68	4.53	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	2.91	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.79	2.61	2.41
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.69	2.51	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.28	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.20	2.01	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.18	1.98	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.16	1.96	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.12	1.91	1.66
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.00	1.79	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	1.92	1.70	1.39
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	1.88	1.65	1.33
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.85	1.63	1.28
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.83	1.61	1.26
∞	3.84	3.00	2.61	2.37	2.22	2.10	2.01	1.94	1.75	1.52	1.00

Figure 6: Statistical table for $F(m, n)$ at the 5% level.

Critical Values of the F Distribution
(α = .01)

df within	df between										
	1	2	3	4	5	6	7	8	12	24	∞
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	9.89	9.47	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.72	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.71	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	3.96	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	3.80	3.43	3.01
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.67	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.46	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.17	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	2.99	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	2.90	2.52	2.07
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	2.87	2.49	2.04
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.66	2.29	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.50	2.12	1.60
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.42	2.03	1.50
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.37	1.98	1.43
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.34	1.95	1.38
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.19	1.79	1.00

Figure 7: Statistical table for $F(m, n)$ at the 1% level.

Critical Values of the χ^2 Distribution

df	Area in the Upper Tail					
	0.99	0.95	0.9	0.1	**0.05**	**0.01**
1	0.000	0.004	0.016	2.706	3.841	6.635
2	0.020	0.103	0.211	4.605	5.991	9.210
3	0.115	0.352	0.584	6.251	7.815	11.345
4	0.297	0.711	1.064	7.779	9.488	13.277
5	0.554	1.145	1.610	9.236	11.070	15.086
6	0.872	1.635	2.204	10.645	12.592	16.812
7	1.239	2.167	2.833	12.017	14.067	18.475
8	1.646	2.733	3.490	13.362	15.507	20.090
9	2.088	3.325	4.168	14.684	16.919	21.666
10	2.558	3.940	4.865	15.987	18.307	23.209
11	3.053	4.575	5.578	17.275	19.675	24.725
12	3.571	5.226	6.304	18.549	21.026	26.217
13	4.107	5.892	7.042	19.812	22.362	27.688
14	4.660	6.571	7.790	21.064	23.685	29.141
15	5.229	7.261	8.547	22.307	24.996	30.578
16	5.812	7.962	9.312	23.542	26.296	32.000
17	6.408	8.672	10.085	24.769	27.587	33.409
18	7.015	9.390	10.865	25.989	28.869	34.805
19	7.633	10.117	11.651	27.204	30.144	36.191
20	8.260	10.851	12.443	28.412	31.410	37.566
21	8.897	11.591	13.240	29.615	32.671	38.932
22	9.542	12.338	14.041	30.813	33.924	40.289
23	10.196	13.091	14.848	32.007	35.172	41.638
24	10.856	13.848	15.659	33.196	36.415	42.980
25	11.524	14.611	16.473	34.382	37.652	44.314

Figure 8: Statistical table for $\chi^2(q)$.